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On Computing the Number of Latin Rectangles

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Abstract Doyle (circa 1980) found a formula for the number of $k \times n$ Latin rectangles $L_{k,n}$. This formula remained dormant until it was recently used for counting $k \times n$ Latin rectangles, where $k \in \{4, 5, 6\}$. We give a formal proof of Doyle's formula for arbitrary k. We also improve a previous implementation of this formula, which we use to find $L_{k,n}$ when $k = 4$ and $n \leq 150$, when $k = 5$ and $n \leq 40$ and when $k = 6$ and $n \leq 15$. Motivated by computational data for $3 \leq k \leq 6$, some research problems and conjectures about the divisors of $L_{k,n}$ are presented.

Keywords Latin square · Latin rectangle

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1 Introduction

A $k \times n$ *Latin rectangle* is a $k \times n$ matrix L, with symbols from $\{1, 2, \ldots, n\}$, such that each row and each column contains only distinct symbols. If $k = n$ then L is a *Latin square* of order n. Let $L_{k,n}$ be the number of $k \times n$ Latin rectangles.

The main purpose of this article is to report the computation of previously unknown values of $L_{k,n}$ (for $4 \le k \le 6$). To the end, we use a formula by Doyle, for which we give a proof for in Sect. 3. We use these values to provide evidentiary support for a question we pose regarding the divisors of $L_{k,n}$ (Question 4). We also present conjectures about divisors of $K_{3,n}$ (Conjecture 5). Questions about divisors of $L_{n,n}$ were previously raised by Alter [1] and about divisors of $L_{k,n}$ in [35].

A Latin rectangle is called *normalised* if the first row is $(1, 2, \ldots, n)$, and *reduced* if the first row is $(1, 2, \ldots, n)$ and the first column is $(1, 2, \ldots, k)^T$. Let $K_{k,n}$ denote the number of normalised $k \times n$ Latin rectangles and let $R_{k,n}$ denote the number of reduced $k \times n$ Latin rectangles. In the case of Latin squares, the numbers $L_{n,n}$, $K_{n,n}$ and $R_{n,n}$ will be denoted L_n, K_n and R_n, respectively. The three numbers $L_{k,n}$, $K_{k,n}$ and $R_{k,n}$ are related by

$$L_{k,n} = n!K_{k,n} = \frac{n!(n-1)!}{(n-k)!}R_{k,n}.$$

In particular

$$L_n = n!K_n = n!(n-1)!R_n.$$

So finding the value of $L_{k,n}$ is essentially the same as finding the value of $R_{k,n}$ or $K_{k,n}$. McKay and Wanless [23] published a table of values for $R_{k,n}$ when $2 \le k \le n \le 11$, which were obtained by lengthy computer enumerations. Some values of $R_{k,n}$ for $k \in \{4, 5, 6\}$ were reported in [36]. Also see [36] for a survey of the formulae involving the number of Latin rectangles.

Gessel [15] proved that for every $k \ge 1$, there exists a finite $M = M(k)$ such that there exists $M + 1$ polynomials $c_i(n)$ such that

$$\sum_{0 \le i \le M} c_i(n) L_{k,n+i} = 0$$

for all $n \ge k$ (other than when each $c_i(n) = 0$). So recurrence relations theoretically exist for $L_{k,n}$ for any fixed k. For example, we know that

$$(n+1)L_{1,n} - L_{1,n+1} = 0$$

since $L_{1,n} = n!$ and

$$(n+2)(n+1)^2 L_{2,n} + (n+2)(n+1)L_{2,n+1} - L_{2,n+2} = 0$$

since $L_{2,n} = n!d_n$ where d_n is the number of derangements of n elements, and d_n satisfies the well-known recurrence $d_n = (n-1)(d_{n-1} + d_{n-2})$ where $d_0 = 1$ and $d_1 = 0$. Theorem 1 (in the next section) gives such a recurrence for $k = 3$.

In the next section we give an historical survey of the formulae for $R_{3,n}$, the case of three-line Latin rectangles. Despite the numerous formulae for $R_{3,n}$, we are still unable to fully explain the growth in the largest power of 2 that divides $R_{3,n}$. Afterwards, we use a formula by Doyle (improving on the implementation used in [35]) to compute values of $R_{k,n}$ when $4 \leq k \leq 6$. We give a proof of this formula in Sect. 3. We find that the divisors of $R_{k,n}$ with $k \in \{4, 5, 6\}$ display similar behaviour to that of $R_{3,n}$. Based on this data, in Sect. 5 we discuss the asymptotic behaviour of the divisors of $R_{k,n}$ for fixed k and present some conjectures about the divisors of $R_{3,n}$.

2 Three-line Latin Rectangles

The enumeration of three-line Latin rectangles has a long history, which we will now review in detail. We also direct the reader to [36], which gives a survey of formulae for $L_{k,n}$ for general k, but does not go into detail about the three-line case. We list the first few non-zero values of $L_{3,n}$, $K_{3,n}$ and $R_{3,n}$ in Table 1.

There are many known general formulae for $L_{k,n}$ (see [36]), of which $L_{3,n}$ is a special case. For example, the following result originated with MacMahon [22] in 1898. Let $X = (x_{ij})$ be a $k \times n$ matrix whose symbols are the kn variables x_{ij}. We index the rows of X by $[k] := \{1, 2, \ldots, k\}$ and the columns of X by $[n] := \{1, 2, \ldots, n\}$. Let $\mathcal{S}_{k,n}$ be the set of injections $\sigma : [k] \to [n]$. We define the *permanent* of the rectangular matrix X to be

$$\mathrm{per}(X) = \sum_{\sigma \in \mathcal{S}_{k,n}} \prod_{i=1}^{k} x_{i\sigma(i)}.$$

Then $L_{k,n}$ is the coefficient of $\prod_{i=1}^{k} \prod_{j=1}^{n} x_{ij}$ in $\mathrm{per}(X)^n$. Let $\mathcal{B}_{k,n}$ be the set of $k \times n$ $(0, 1)$-matrices. Fu [13] and Shao and Wei [31] gave the formula

$$L_{k,n} = \sum_{A \in \mathcal{B}_{k,n}} (-1)^{\sigma_0(A)} \mathrm{per}(A)^n,$$

where $\sigma_0(A)$ is the number of 0 elements in A, which was generalised in [36] (see also [23]).

The first published formula specifically for $L_{3,n}$ seems to be by Jacob [19] in 1930, although some errors in [19] were later rectified by Kerawala [20] to give the following theorem.

Table 1 $L_{3,n}$, $K_{3,n}$ and $R_{3,n}$ for small n

n	3	4	5	6	7	8	Sloane [34] ref.
$L_{3,n}$	12	576	66240	15321600	5411750400	2834466324480	
$K_{3,n}$	2	24	552	21280	1073760	70299264	A000186
$R_{3,n}$	1	4	46	1064	35792	1673792	A001623

Theorem 1 *For any $n \geq 3$,*

$$\sum_{0 \leq i \leq 5} c_i(n) K_{3,n+i} = 0$$

where

$$\begin{aligned} c_0(n) &= -4(n+1)(n+2)(n+3)(n+4)^2, \\ c_1(n) &= 2(n+2)(n+3)(n+4)(n^2+5n+3), \\ c_2(n) &= (n+3)(n+4)(n^2+8n+13), \\ c_3(n) &= (n+3)(n+4)(n^2+8n+17), \\ c_4(n) &= (n+4)(n^2+8n+17), \\ c_5(n) &= -(n+3). \end{aligned}$$

Riordan [26] proved that $K_{3,n} \sim n!^2 \exp(-3)$, which he described as "a result which Kerawala surmised but failed to prove, though his numerical evidence was practically conclusive (agreeing with $\exp(-3)$ to seven decimal places)." Jacob [19] also mentioned the possibility of $n!^2/K_{3,n} \to \exp(3)$, but it was subsequently abandoned. This asymptotic formula was generalised by Erdős and Kaplansky [11], who found that

$$L_{k,n} \sim n!^k \exp\big(-k(k-1)/2\big)$$

for $k = O\big((\log n)^{3/2-\epsilon}\big)$, which has since been extended (see [36] for an history). The most up-to-date asymptotic enumeration is by Godsil and McKay [17], who proved

$$L_{k,n} \sim n!^k \Big(n(n-1)\cdots(n-k+1)/n^k\Big)^n (1-k/n)^{-n/2} \exp(-k/2)$$

as $n \to \infty$ with $k = o(n^{6/7})$.

In [27] (see also [29]), Riordan gave a formula for $K_{3,n}$ that involves the *problème de ménages*. Specifically

$$K_{3,n} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{i} d_i d_{n-i} u_{n-2i} \tag{1}$$

where d_n is the number of derangements of n elements (Sloane's [34] A000166) and u_n is given by

$$u_n = \sum_{i=0}^{n} (-1)^i \frac{2n}{2n-i} \binom{2n-i}{i} (n-i)!, \tag{2}$$

where $u_0 = 1$. The numbers u_n for $n \geq 2$ are the *ménages numbers* (Sloane's A000179). The ménages numbers can also be defined as the number of permutations σ of $[n]$ such that $\sigma(i) \not\equiv i \pmod{n}$ and $\sigma(i) \not\equiv i+1 \pmod{n}$ for all $i \in [n]$. However, for (1) to be valid, we require $u_1 = -1$, which is inconsistent with the "number of σ" definition, but is consistent with (2) (this issue was raised by Vladimir Shevelev on Sloane's A000186). Riordan's formula (1) was generalised by Moser [24] to count normalised three-line Latin rectangles where the derangement defined by the second row does not have cycle lengths belonging to some set S. A related generalisation was given by Shevelev [32] where instead the cycle lengths belong to S.

Riordan [28] also gave the recurrence equation

$$K_{3,n} = n^2 K_{3,n-1} + n(n-1)K_{3,n-2} + 2n(n-1)(n-2)K_{3,n-3} + s_n \tag{3}$$

where

$$s_n = -ns_{n-1} - (n-1)2^n \tag{4}$$

and $s_0 = 1$. We find that using (3)–(4) is the fastest way to compute $K_{3,n}$ in practice. This is unsurprising as it requires only a finite number of arithmetic operations to compute $K_{k,n}$ (from previous terms and the auxiliary function s_n). With the exception of Kerawala's recurrence (Theorem 1), all other formulae for $L_{k,n}$ we survey require a number of arithmetic operations which grows with n.

Dulmage [9] posed an "explicit (though complicated)" formula for $L_{3,n}$ as a problem, which was later refined by Dulmage and McMaster [10] who gave

$$K_{3,n} = \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{i=0}^{n-2j} (-1)^{j+i} \binom{n}{j,\, j,\, i} a_{j+i,i}\, a^2_{n-j-i,n-2j-i},$$

where

$$a_{n,p} = \sum_{i=0}^{p} (-1)^i \binom{p}{i} (n-i)!,$$

which they used to compute $K_{3,n}$ for $n \le 40$ on an IBM 370/158 in approximately five seconds.

Bogart and Longyear [3] gave

$$L_{3,n} = n!^2 \sum_{p+q+r+s+d=n} \frac{(-1)^{p+q+r}\, 2^s\, (p+d)!\,(q+d)!\,(r+d)!}{d!^2\, p!\, q!\, r!\, s!}$$

where the sum is over all non-negative integers p, q, r, s and d which sum to n. Bogart and Longyear used their formula to find $L_{3,n}$ for $n \le 11$ (although some typographical errors were pointed out in [39]) and gave an approximation of $L_{3,n}$ for $n \le 20$.

Riordan [28] said Yamamoto [44] found the equation

$$R_{3,n} = \sum_{a+b+c=n} n(n-3)!(-1)^b \frac{2^c a!}{c!} \binom{3a+b+2}{b},$$

where the sum is over all non-negative integers a, b and c which sum to n.

Goulden and Jackson [18] showed that $K_{3,n}$ is the coefficient of $x^n/n!$ in the series expansion of

$$e^{2x}\sum_{i\geq 0}\frac{x^i}{(1+x)^{3i+3}}.$$

This was generalised by Gessel [14] to count pairs of discordant derangements. Specifically, the number of pairs (π,σ) of derangements of $[n]$ such that $\pi\sigma^{-1}$ is also a derangement, π has a cycles and σ has b cycles is the coefficient of $\alpha^a\beta^b x^n$ in

$$e^{2\alpha\beta x}\sum_{i\geq 0}\frac{(\alpha)_i(\beta)_i}{i!}\frac{x^i}{(1+\alpha x)^{i+\beta}(1+\beta x)^{i+\alpha}(1+x)^{n+\alpha\beta}}$$

where $(\alpha)_i=\alpha(\alpha+1)\cdots(\alpha+i-1)$.

Riordan [28] gave the recurrence congruence $R_{3,n+p}\equiv 2R_{3,n} \pmod p$ for all odd primes p, provided $n\geq 3$, which was generalised by Carlitz [5] to $R_{3,n+t}\equiv 2^t R_{3,n} \pmod t$ for all $t\geq 1$. This was generalised to k-line Latin rectangles in [39], who gave

$$R_{k,n+t}\equiv\left((-1)^{k-1}(k-1)!\right)^t R_{k,n} \pmod t \tag{5}$$

for all $k\geq 1$ and $t\geq 1$ provided $n\geq k$. It was also noted in [39] that some primes p do not divide $R_{3,n}$ for any $n\geq 3$. For example, any prime

$$p\in\{3,5,11,29,37,41,43,53,67,79,83,97\}$$

does not divide $R_{3,n}$ for all $n\geq 3$ (see [36,39]).

There are also published formulae for four-line Latin rectangles by Light Jr. [21], Pranesachar et al. [2,25] and Doyle [6]. We will discuss Doyle's formula in the next section, which is, by far, the best method for finding the exact value of $R_{k,n}$ for $4\leq k\leq 6$.

3 Doyle's Formula

The computation of $L_{k,n}$ for $k\in\{1,2,3\}$ can be considered as effectively "solved": we have $L_{1,n}=n!$, and $L_{2,n}$ can be found by computing the number of derangements d_n. For $n=3$, using Riordan's recurrence (i.e., (3)–(4)), we computed $L_{3,n}$ for all $n\leq 2^{21}\approx 2\times 10^6$ on a desktop computer in under 38 hours (we use these results in Section 5). As we will see, computing $L_{k,n}$ for $k\geq 4$ is much more difficult practically.

Exact enumeration is difficult for $k>3$. – Skau [33]

In [6], Doyle gave formulae for $K_{k,n}$ for $k\leq 4$ and indicated how they could be generalised to arbitrary k. This generalisation was subsequently used in [36] to find values $K_{k,n}$ when $k\leq 6$ (although a proof was not given). For the sake of rigour, we will give a proof of Doyle's generalised formula. The overall idea of the formula is to use Inclusion-Exclusion on ordered n-tuples of columns. We include all such

n-tuples, then exclude those that clash. To count the number of n-tuples that clash, Möbius Inversion is used on each column, counting the number of arrangements that "avoid" a certain substructure (e.g. symbol 1 in row 2).

Let $\mathcal{R}$ be the set of non-negative integer vectors $\mathbf{s} = (s_i)_{1\le i\le 2^{k-1}}$ such that $\sum_i s_i = n$. For $1 \le i \le 2^{k-1}$, let $\Delta_i = (\delta_{ij})_{1\le j\le 2^{k-1}}$, where δ_{ij} is the Kronecker δ-function. For any non-negative integer i let $b_j(i)$ be the j-th binary digit of i, for example $\left(b_j(13)\right)_{j\ge 1} = (1, 0, 1, 1, 0, 0, \ldots)$. Let $||\mathbf{s}|| = \sum_{1\le i\le 2^{k-1}} \sum_{1\le j\le k-1} s_i b_j(i)$.

Theorem 2

$$K_{k,n} = \sum_{\mathbf{s}\in\mathcal{R}} (-1)^{||\mathbf{s}||} \binom{n}{s_1, s_2, \ldots, s_{2^{k-1}}} \prod_{i=1}^{2^{k-1}} g\left(\mathbf{s} - \Delta_i\right)^{s_i} \tag{6}$$

where for $\mathbf{a} = (a_1, a_2, \ldots, a_{2^{k-1}})$,

$$g(\mathbf{a}) = \sum_{P\in\mathcal{P}_{k-1}} \prod_{p\in P} (-1)^{|p|-1}(|p|-1)!\, f_p(\mathbf{a}) \tag{7}$$

where $\mathcal{P}_{k-1}$ *is the set of partitions of* $\{1, 2, \ldots, k-1\}$ *and*

$$f_p(\mathbf{a}) = \sum_{i: b_j(i)=0 \forall j\in p} a_i \tag{8}$$

for all $p \subseteq \{1, 2, \ldots, k-1\}$.

Proof Let $\mathcal{S}$ be the set of $n \times (k-1)$ $(0, 1)$-matrices and let $D = (d_{ij}) \in \mathcal{S}$. We say a matrix $D' = (d'_{ij}) \in \mathcal{S}$ *avoids* D if $d'_{ij} = 0$ whenever $d_{ij} = 1$. Let

- $Z = Z(D) = \{D' \in \mathcal{S} : D' \text{ avoids } D\}$,
- $X = X(D) = \{D' \in Z :$ every column of D' contains a unique 1 and every row of D' contains at most one 1$\}$ and
- $Y_P = Y_P(D) = \{D' \in Z :$ every column of D' contains a unique 1 and columns c and c' of D' are identical if there exists a part $p \in P$ for which both $c, c' \in p\}$ for any partition $P \in \mathcal{P}_{k-1}$.

Let $L = (l_{ij})$ be a normalised $k \times n$ Latin rectangle. From L we can construct a $k \times n \times n$ $(0, 1)$-array $P = (p_{ijr})$ where $p_{ijr} = 1$ whenever $l_{ij} = r$. We will find it helpful to think of P as an ordered n-tuple of $n \times (k-1)$ $(0, 1)$-matrices $(M_1, M_2, \ldots, M_n) \in X^n$ where the (r, i)-th coordinate of M_j is $p_{(i+1)jr}$. An example of this equivalence is given in Fig. 1; the shaded entries must be zero since L is normalised.

For $1 \le j \le n$, and $D \in \mathcal{S}$, let D_j be the matrix formed from D after the j-th row has been converted to a row of ones. By Inclusion-Exclusion

$$K_{k,n} = \sum_{D\in\mathcal{S}} (-1)^{\sigma(D)} |X(D_1)| \cdot |X(D_2)| \cdots |X(D_n)| \tag{9}$$

$$L = \begin{array}{|cccc|}\hline 1&2&3&4\\ 3&1&4&2\\ 2&4&1&3\\ \hline\end{array} \quad \leftrightarrow \quad M_1 = \begin{array}{|cc|}\hline 0&0\\ 0&1\\ 1&0\\ 0&0\\ \hline\end{array},\ M_2 = \begin{array}{|cc|}\hline 1&0\\ 0&0\\ 0&0\\ 0&1\\ \hline\end{array},\ M_3 = \begin{array}{|cc|}\hline 0&1\\ 0&0\\ 0&0\\ 1&0\\ \hline\end{array},\ M_4 = \begin{array}{|cc|}\hline 0&0\\ 1&0\\ 0&1\\ 0&0\\ \hline\end{array}$$

Fig. 1 Converting between L and $M_1, M_2, \ldots, M_n$

$$D = \begin{array}{|cc|}\hline 1&1\\ 0&1\\ 1&0\\ 0&0\\ 0&0\\ 0&1\\ 0&0\\ 1&1\\ 0&0\\ \hline\end{array} \leftrightarrow \begin{array}{c} 11\\ 01\\ 10\\ 00\\ 00\\ 01\\ 00\\ 11\\ 00 \end{array} \leftrightarrow \begin{array}{c} 3\\ 1\\ 2\\ 0\\ 0\\ 1\\ 0\\ 3\\ 0 \end{array} \quad \rightarrow \quad \begin{array}{|rc|}\hline s_1 & 2\\ s_2 & 1\\ s_3 & 2\\ s_4 = s_0 & 4\\ \hline\end{array}$$

Fig. 2 Finding **s** from D

where $\sigma(D)$ is the number of ones in D. The Inclusion-Exclusion works as follows. We include all the n-tuples $(M_1, M_2, \ldots, M_n) \in X^n$ for which the j-th row of each M_j does not contain a 1. Suppose $D \in \mathcal{S}$ where the (r, i)-th element of D is 1. Suppose also that we attempt to construct a normalised Latin rectangle L from $(M_1, M_2, \ldots, M_n) \in X^n$ in which every M_j avoids D. Then this would imply that symbol r does not appear in row $i+1$ of L, giving rise to a clash. Therefore, we then exclude all such n-tuples for which there exists some $D \in \mathcal{S}$ containing a 1 for which every M_j avoids D.

From any $D \in \mathcal{S}$ we construct a vector $\mathbf{s} = (s_1, s_2, \ldots, s_{2^{k-1}})$ in the following way. Each row of D can be considered the binary digits of some number between 0 and $2^{k-1} - 1$ (inclusive). Let s_t be the number of rows in D that are the binary digits of t. To be consistent with [36], we use $s_{2^{k-1}} = s_0$ in the statement of the theorem. This does not affect the proof since the first $k-1$ binary digits of both 0 and 2^{k-1} are zero. Hence $\sigma(D) = ||\mathbf{s}||$. This process is depicted in an example in Fig. 2.

The number of $D \in \mathcal{S}$ that give rise to a given $\mathbf{s} \in \mathcal{R}$ is given by the multinomial coefficient

$$\binom{n}{s_1, s_2, \ldots, s_{2^{k-1}}}.$$

We can partition $\mathcal{S}$ according to $\mathbf{s} \in \mathcal{R}$, whence two matrices D and D' in the same part have $|X(D)| = |X(D')|$. For each $\mathbf{s} \in \mathcal{R}$, choose a representative $D^* = D^*(\mathbf{s})$. Hence (9) becomes

$$K_{k,n} = \sum_{\mathbf{s} \in \mathcal{R}} (-1)^{||\mathbf{s}||} \binom{n}{s_1, s_2, \ldots, s_{2^{k-1}}} |X(D_1^*)| \cdot |X(D_2^*)| \cdots |X(D_n^*)|. \tag{10}$$

If we define $g(\mathbf{a}) = |X(D^*(\mathbf{a}))|$, then (10) becomes

$$K_{k,n} = \sum_{\mathbf{s} \in \mathcal{R}} (-1)^{||\mathbf{s}||} \binom{n}{s_1, s_2, \ldots, s_{2^{k-1}}} \prod_{i=1}^{2^{k-1}} g\big(\mathbf{s} - \Delta_i + \Delta_{2^{k-1}-1}\big)^{s_i} \tag{11}$$

Fig. 3 An example of an $N \in Y_P(D)$ where $P = \{\{1, 2\}, \{3\}\}$

$$D = \begin{array}{|ccc|} 0&0&0\\ 0&1&0\\ 1&1&0\\ 0&1&0\\ 0&0&1\\ 0&0&0\\ 0&1&0 \end{array} \qquad N = \begin{array}{|cc|c|} 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 1&1&0\\ 0&0&1\\ 0&0&0 \end{array} \leftrightarrow N_{\{1,2\}} = \begin{array}{|cc|} 0&0\\ 0&0\\ 0&0\\ 0&0\\ 1&1\\ 0&0\\ 0&0 \end{array},\ N_{\{3\}} = \begin{array}{|c|} 0\\ 0\\ 0\\ 0\\ 0\\ 1\\ 0 \end{array}$$

since if two rows i and i' of D^* are the same, then $|X(D_i^*)| = |X(D_{i'}^*)|$. The first $k-1$ binary digits of $2^{k-1} - 1$ are all ones, so we will later delete $\Delta_{2^{k-1}-1}$ from (11) since it makes no difference to (7) and (8). It is now sufficient to prove (7).

We can decompose any $N \in Y_P$ into $|P|$ submatrices N_p for each $p \in P$. An example of this decomposition is given in Fig. 3; we shade the entries that N and the N_p avoid. For a given $D \in \mathcal{S}$, the number of possible submatrices N_p is equal to the number of rows of D that map to a number t for which the j-th binary digit of t is zero for all $j \in p$ (informally, we can choose a single row of ones in N_p that avoids the ones in D); this number is given by (8). Therefore $|Y_P| = \prod_{p\in P} f_p(\mathbf{a})$. It is now sufficient to show that $g(\mathbf{a}) = \sum_{P\in\mathcal{P}_{k-1}} |Y_P| \prod_{p\in P}(-1)^{|p|-1}(|p|-1)!$.

If $P, Q \in \mathcal{P}_{k-1}$ and P is a refinement of Q, then we write $P \trianglelefteq Q$ and $Q \trianglerighteq P$. If additionally $P \neq Q$, then we write $P \triangleleft Q$ and $Q \triangleright P$. For the remainder of this proof, $P \in \mathcal{P}_{k-1}$ will be the partition of cardinality $k-1$, i.e., we will henceforth assume $P = \{\{1\}, \{2\}, \ldots, \{k-1\}\}$.

By Inclusion-Exclusion,

$$|X| = |Y_P| - \Big|\bigcup_{Q \triangleright P} Y_Q\Big| = \sum_{Q \trianglerighteq P} \mu(P, Q)|Y_Q|$$

for some integer coefficients $\mu(P, Q)$, where $\mu(P, P) = 1$. To ensure the matrices $M \in Y_P$ with some duplicated columns are counted 0 times overall, the coefficients $\mu(P, Q)$ must satisfy $\sum_{P\trianglelefteq R\trianglelefteq Q} \mu(P, R) = 0$, whenever $P \triangleleft Q$. Therefore $\mu(P, Q) = -\sum_{P\trianglelefteq R\triangleleft Q} \mu(P, R)$. Hence μ is the Möbius Function for the lattice of partitions $\mathcal{P}_{k-1}$ (see [30, pp. 359–360] or [43, pp. 333–336] for example). Since P is the partition of cardinality $k-1$, we find $\mu(P, Q) = (-1)^{k-1-|Q|} \prod_{q\in Q}(|q|-1)! = \prod_{q\in Q}(-1)^{|q|-1}(|q|-1)!$ □

Doyle's equation for $K_{k,n}$ might seem intimidating at first, but if we fix a value of k, then $g(\mathbf{a})$ and the $f_p(\mathbf{a})$ are fixed multivariate polynomials. Afterwards, for each vector $\mathbf{s} \in \mathcal{R}$, computing its contribution to (6) is a straightforward task.

Doyle's formula was used in [36] to find $R_{4,n}$ for $n \leq 80$, $R_{5,n}$ for $n \leq 28$ and $R_{6,n}$ for $n \leq 13$. Motivation for finding more of these numbers comes from the following question in [35], which was posed after considering the data for $R_{4,n}$ and $R_{5,n}$ in [36].

Question 3 *How do the prime power divisors p^a of $R_{k,n}$ behave asymptotically for a fixed $k > p$ as $n \to \infty$ or as both $k \to \infty$ and $n \to \infty$?*

Some general results concerning the divisors of $R_{k,n}$ were given by [39]. For example, we know that $p^{\lfloor n/p \rfloor}$ divides $R_{k,n+d}$ and $K_{k,n}$ when p is a prime and $d \geq k > p$.

4 Implementation

The first author gave a basic implementation of Doyle's formula in C available from

$$\text{http://code.google.com/p/latinrectangles/downloads/list} \tag{12}$$

which was used in [36] to find $R_{4,n}$, $R_{5,n}$ and $R_{6,n}$ for $n \leq 80$, $n \leq 28$ and $n \leq 13$, respectively. Some basic, but significant improvements to the code were made, most notably:

1. *Parallelisation.* In order to implement Doyle's formula in parallel, we partition $\mathcal{R}$ into x parts, where x is the number of processes to be run in parallel. Using lexicographic ordering on $\mathcal{R}$, we find $x-1$ vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{x-1}$, such that the x sets

$$\begin{aligned}
&\{\mathbf{s} \in \mathcal{R} : \mathbf{s} < \mathbf{z}_1\}, \\
&\{\mathbf{s} \in \mathcal{R} : \mathbf{z}_i \leq \mathbf{s} < \mathbf{z}_{i+1}\} \qquad \text{for } 1 \leq i \leq x-2, \\
&\{\mathbf{s} \in \mathcal{R} : \mathbf{z}_{x-1} \leq \mathbf{s}\}
\end{aligned}$$

   have roughly the same cardinality. The Maple code used to find the vectors $\mathbf{z}_i$ is also available from (12).
2. *Symmetry.* We identify a group G that acts on $\mathcal{R}$ such that the contribution to (6) is invariant. For $k \in \{4, 5, 6\}$, the order of this group is $|G| = (k-1)!$. If $G(\mathbf{s})$ denotes the orbit of $\mathbf{s} \in \mathcal{R}$, then we only include the contribution of $\mathbf{s}$ to (6) whenever $\mathbf{s} = \min G(\mathbf{s})$, but multiplied by $|G(\mathbf{s})|$. While useful, exploiting this symmetry does not reduce run-time by a factor of $|G|$ since (a) there is additional overhead and (b) sometimes $|G(\mathbf{s})| < |G|$.

Using the aforementioned improvements, we were able to find $R_{k,n}$ when

- $k = 4$ and $n \leq 150$, as at Sloane's [34] A000573,
- $k = 5$ and $n \leq 40$, as in Fig. 4 (Fig. 5 gives the prime factorisations),
- $k = 6$ and $n \leq 15$, as in Fig. 6.

The improved version of the code can also be downloaded from (12).

Theoretically, if we found enough values of $R_{4,n}$, we could use Sister Celine's technique [45] to propose a recurrence for $R_{4,n}$. However, it seems that very many values of $R_{4,n}$ would be required to find the recurrence, well beyond the values for $n \leq 150$ we currently have.

We can use (5) to check that the values are indeed correct. It is highly unlikely that an erroneous implementation would still return numbers that satisfy all of the recurrences implied by (5). We can also check that the number of digits is roughly what we expect based on the asymptotic formula and neighbouring data points.

Doyle's formula iterates through each $\mathbf{s} \in \mathcal{R} = \mathcal{R}(k, n)$; a "stars and bars" argument gives

$$|\mathcal{R}(k, n)| = \binom{n + 2^{k-1} - 1}{2^{k-1} - 1}.$$

n	$R_{5,n}$
5	56
6	9408
7	11270400
8	27206658048
9	112681643083776
10	746988383076286464
11	7533492323047902093312
12	111048869433803210653040640
13	2315236533572491933131807916032
14	66415035616070432053233927044726784
15	2560483881619577552584872021599994249216
16	130003705747573381528820187969499352864391168
17	8540614065591861115863858023929942463204158341120
18	714772705022049580010386905464376609190681339062386688
19	75163163562802272546579759450749095599610461567358920032256
20	9809720003910626776223482379751753587443069548693920857303547904
21	1571535264701285629600025091867663915099001357016958197822862919729152
22	305967368069117220345719015650882351240204884316352710461216388953743032320
23	71742822040206698482547032648440680248173149276783605396347465027480511202721792
24	20093299726164942410036767774030176748339141446536947374523570181642887594307280175104
25	6671363422740192076170128383025874322430996291082893578356639976639833297028025599106482176
26	2607692996972092696824065181857133325599861640770239539217211513790688430496101840933225630793728
27	1192341043681765508107725478415585115424290311661367053795101824269858341198689704358509964052964311040
28	633985642868552678440810273027244480070214946032409979061521248934113017591003027196021410395800315287830528
29	389863266865164324387916378672365140155984577240419607973327761543751643778618720299309101909638377223816673755136
30	275857652903657751160600814537727822159682792695918588105904040061609342339707690900564049424935568511248228465502060544
31	223528711229533750131579209432053914463907715250894091913905482729923954445631366223683361287208873647394485447649925049352192
32	206503802643599790612955830549361333665058006516534461250529321447916066199712592957821804969954985277518619445467493869212716236800
33	216602010320400171752501385822631059980277106402589793768104148179208971708900795896180092675341977036399897246578327576759263198919524352
34	256945424857738894606189218304505951512111882837615637741647954302109498752739528688597729441887484475804705562130347540523813378511761282433024
35	343454085793901415670382700651413829033841459314644482527081222485405557889266573002461717464243880921591447640401097167569367307049389628024669863936
36	515517774955992320855323040363644410076277205040218722138931631209826553990384304788725859345549047506846736705105338690238149639704217358823982557023961088
37	866060210336618414337292610681957242571690190219289656099965664246989988347710950509344968640973665076289218246975484498214468517829727246001241462127789772636160
38	1623461363470184043991363294408631082728889510323172023141923435363881154965071973552177058282895064846701912512609922352171552940546374024648570272353624276399521005568
39	3385763508814815997534414684057743377885030461377799199401969394102590364055427523494741397566971505263112777996060632173683014873359249702966678235639878696802414102803644416
40	7834103318049771485000779095696587381945742974944485999359098670413812198591300962633305392150020926992120278871845946924682925988405332343236393165124176896359208459542537525264384

Fig. 4 $R_{5,n}$ for small n

n	$R_{5,n}$
5	$2 \cdot 7$
6	$2 \cdot 3 \cdot 7^2$
7	$2 \cdot 3 \cdot 5 \cdot 587$
8	$2^{11} \cdot 3 \cdot 23 \cdot 192529$
9	$2^{11} \cdot 3 \cdot 13 \cdot 52251029$
10	$2^{16} \cdot 3 \cdot 19 \cdot 97 \cdot 8483617$
11	$2^{13} \cdot 3 \cdot 29 \cdot 168293 \cdot 20936295857$
12	$2^{17} \cdot 3 \cdot 5 \cdot 7 \cdot 47 \cdot 59 \cdot 313 \cdot 38257310467$
13	$2^{19} \cdot 3 \cdot 7 \cdot 2336488485157166267205$1
14	$2^{27} \cdot 3 \cdot 101 \cdot 449 \cdot 1039 \cdot 3019 \cdot 22811 \cdot 1882698637$
15	$2^{22} \cdot 3 \cdot 19 \cdot 423843896863 \cdot 34662016427839511$
16	$2^{28} \cdot 3 \cdot 3604099 \cdot 40721862001 \cdot 4526515223205743$
17	$2^{25} \cdot 3 \cdot 5 \cdot 15001087 \cdot 13964976140347893908947110110827$
18	$2^{28} \cdot 3 \cdot 1019173084339 \cdot 237316919875331 \cdot 559319730817259$
19	$2^{28} \cdot 3 \cdot 7 \cdot 47 \cdot 149 \cdot 532451 \cdot 347100904121707 \cdot 4239553164518180468847$7
20	$2^{32} \cdot 3 \cdot 7 \cdot 67 \cdot 163$
21	$2^{33} \cdot 3 \cdot 83 \cdot 281 \cdot 204292081063933 \cdot 585232305196091317767148692734312066$9
22	$2^{36} \cdot 3 \cdot 5 \cdot 13 \cdot 241559 \cdot 12966116042479108099276464512087192923642576306645363$1
23	$2^{39} \cdot 3^{10} \cdot 5407 \cdot 120427 \cdot 901145309 \cdot 376635293602221558326481401187618944977013839$1
24	$2^{41} \cdot 3^{11} \cdot 107 \cdot 739951 \cdot 2418119033203$
25	$2^{41} \cdot 3 \cdot 94513 \cdot 54260027 \cdot 25093654805621$
26	$2^{44} \cdot 3^{10} \cdot 7 \cdot 67933 \cdot 202543723 \cdot 2685265441 \cdot 156723690161879 \cdot 619305034179432354947567439552171321683 81$
27	$2^{43} \cdot 3^{12} \cdot 5 \cdot 7 \cdot 53$
28	$2^{48} \cdot 3^{10} \cdot 17491 \cdot 28001$
29	$2^{49} \cdot 3^{11} \cdot 19 \cdot 5087$
30	$2^{52} \cdot 3^{12} \cdot 556351 \cdot 21784135181$
31	$2^{51} \cdot 3^{11} \cdot 23 \cdot 193156177$
32	$2^{58} \cdot 3^{13} \cdot 5 \cdot 10691 \cdot 23371 \cdot 2252593130439283$
33	$2^{55} \cdot 3^{14} \cdot 7 \cdot 5533397053 \cdot 2672442727811 \cdot 1214274498310769429735706213586604173951066446865812311207141276214292668862394563215351295$1
34	$2^{58} \cdot 3^{15} \cdot 7 \cdot 19 \cdot 31 \cdot 620947027$
35	$2^{62} \cdot 3^{14} \cdot 13 \cdot 61 \cdot 101 \cdot 16316333719771 \cdot 600568926261159442761$7
36	$2^{64} \cdot 3^{17} \cdot 73 \cdot 461 \cdot 6911 \cdot 14083 \cdot 38917 \cdot 623597173664628 29 \cdot 1159609386500505427 \cdot 23477265921294304321382486419630973846526050897854155508708352569442218566559$
37	$2^{65} \cdot 3^{15} \cdot 5$
38	$2^{68} \cdot 3^{15}$
39	$2^{72} \cdot 3^{17} \cdot 271 \cdot 337 \cdot 4093081 \cdot 60833363 \cdot 93489371$
40	$2^{73} \cdot 3^{16} \cdot 7 \cdot 29 \cdot 101 \cdot 941 \cdot 132071 \cdot 530165653$

Fig. 5 Factorisation of $R_{5,n}$ for small n

n	$R_{6,n}$	Factorisation
6	9408	$2 \cdot 3 \cdot 7^2$
7	16942080	$2^{10} \cdot 3 \cdot 5 \cdot 1103$
8	335390189568	$2^{11} \cdot 3 \cdot 7 \cdot 173 \cdot 45077$
9	12952605404381184	$2^{14} \cdot 3 \cdot 3253351007$
10	870735405591003709440	$2^{14} \cdot 3 \cdot 5 \cdot 26053 \cdot 15110358097$
11	9629955237329250515877880	$2^{17} \cdot 3 \cdot 5 \cdot 31 \cdot 2334139 \cdot 225638611943$
12	167907691549259296737250620211120	$2^{17} \cdot 3 \cdot 5 \cdot 131 \cdot 110630813 \cdot 65475601447957$
13	4453330421956050777867897829494620160	$2^{21} \cdot 3 \cdot 5 \cdot 7 \cdot 43331 \cdot 5185904205452446940 7499$
14	174210186305611178933806527744459502780 4160	$2^{21} \cdot 3 \cdot 5 \cdot 9923 \cdot 361387484390839 \cdot 1715907088965739$
15	9785145873148199028198458478282304160111 00160000	$2^{26} \cdot 3 \cdot 5 \cdot 7 \cdot 47 \cdot 251 \cdot 70351 \cdot 1652575202161103085073 3157$

Fig. 6 $R_{6,n}$ for small n

So the number of iterations required to compute $R_{4,150}$, $R_{5,40}$ and $R_{6,15}$ is as given below:

$$|\mathcal{R}(4, 150)| = \binom{157}{7} = 407,340,975,756,$$
$$|\mathcal{R}(5, 40)| = \binom{55}{15} = 11,899,700,525,790,$$
$$|\mathcal{R}(6, 15)| = \binom{46}{31} = 511,738,760,544,$$

although we do reduce the number of required iterations by a factor of $(k-1)!$ through symmetry. For the purpose of inspecting the divisors of $R_{k,n}$, we are particularly interested in finding more values of $R_{5,n}$.

We can estimate how many times longer it would take to compute $R_{k,n+1}$ than $R_{k,n}$ with the ratio

$$r_{k,n} := \frac{|\mathcal{R}(k, n+1)|}{|\mathcal{R}(k, n)|} = \frac{n + 2^{k-1}}{n+1},$$

which is close to 1 when n is much larger than 2^{k-1}. For example, $r_{4,150} \approx 1.1$, $r_{5,40} \approx 1.4$ and $r_{6,15} \approx 2.9$. So it would be possible to find more values of $R_{4,n}$, but we had to stop somewhere.

5 Divisors

The question of divisors of R_n, and in particular powers of 2 and 3, was originally raised by Alter [1] about Latin squares. This question, along with analogous questions for Latin rectangles, has been a hot research topic for Stones [35], and is linked to the autotopisms and automorphisms of Latin rectangles [4,39] (see also [12,38]), and orthomorphisms and partial orthomorphisms of finite cyclic groups [40,41]. Divisors for the number of even/odd Latin squares have been used in proving special cases of the Alon-Tarsi Conjecture [7] (see also [8,16,37,42]).

For any prime p, let $\omega_p(n)$ be the largest non-negative integer such that $p^{\omega_p(n)}$ divides n, that is the p-adic valuation of n. In Fig. 7, we plot $\omega_2(R_{k,n})$ for $k \in \{3, 4, 5, 6\}$ and small n. In Fig. 8, we plot $\omega_3(R_{k,n})$ for $k \in \{4, 5\}$ and small n. This data leads us to the following question.

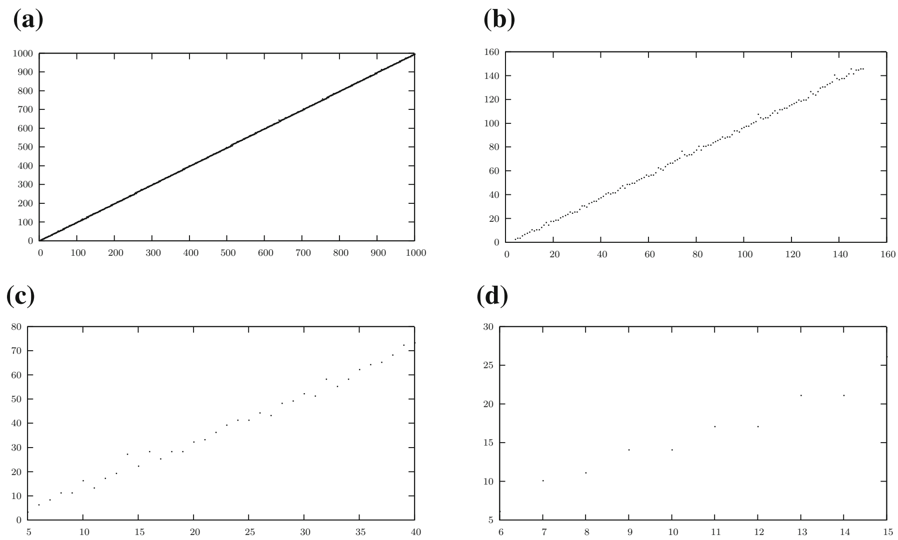

Fig. 7 $\omega_2(R_{k,n})$ for $k \in \{3,4,5,6\}$ and small n **a** $\omega_2(R_{3,n})$ vs. n. **b** $\omega_2(R_{4,n})$ vs. n. **c** $\omega_2(R_{5,n})$ vs. n. **d** $\omega_2(R_{6,n})$ vs. n

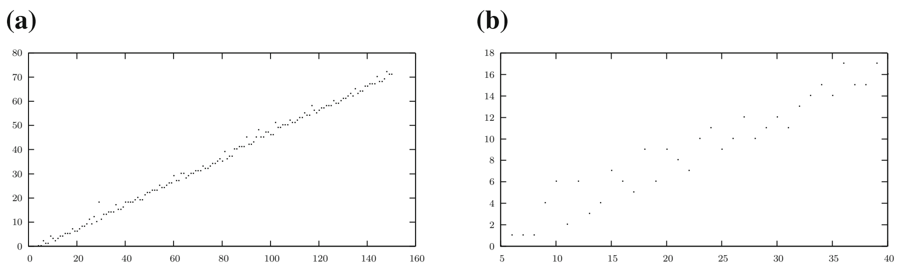

Fig. 8 $\omega_3(R_{k,n})$ for $k \in \{4,5\}$ and small n. **a** $\omega_3(R_{4,n})$ vs. n. **b** $\omega_3(R_{5,n})$ vs. n

Fig. 9 Approximations of the gradients in Figs. 7 and 8

$p = 2$

k	gradient
3	1
4	1
5	2
6	2

$p = 3$

k	gradient
3	0
4	1/2
5	1/2

Question 4 *For any fixed k and fixed prime $p < k$, is*

$$\omega_p(R_{k,n}) \geq \frac{\lfloor (k-1)/p \rfloor}{p-1} n - o(n)?$$

The gradients implied by Question 4 match those in Fig. 9, which are suggested by the empirical data.

From [39], we already know that $\omega_p(R_{k,n+k}) \geq \lfloor n/p \rfloor$ for all primes $p < k$. Furthermore, it was shown in [4] that $\omega_p(R_n) \geq n/(p-1) - O(\log^2 n)$ as $n \to \infty$.

The divisors of $R_{k,n}$ for primes $p \geq k$ do not display the same characteristics as when $p < k$. For example, 3 does not divide $R_{3,n}$ for any $n \geq 3$, whereas a growing power of 3 divides $R_{4,n}$ [39].

In the case of three-line Latin rectangles, Riordan's recurrence (3) gives us the ability to quickly compute values of $K_{3,n}$. We will now list some conjectures for three-line Latin rectangles that are motivated by these numbers.

Conjecture 5 *1.* $\omega_2(K_{3,n}) = \omega_2(n!) = n - 1$ *whenever* $n = 2^i$ *and* $i \geq 2$. *(True for* $i \leq 21$.*)*

2. $\omega_2(K_{3,n}) \geq \omega_2(n!)$ *for all* $n \geq 3$. *(True for* $n \leq 2^{18}$.*)*

3. $\omega_2(K_{3,n}) = n - i + 1$ *whenever* $n = 2^i - 3$ *and* $i \geq 3$. *(True for* $i \leq 21$.*)*

Kerawala's recurrence (Theorem 1) suggests that $\omega_2(K_{3,n})$ might be less than usual when n has the form $2^i - 3$, which is what we see in the numerical data.

Conjecture 5 implies that $\omega_2(K_{3,n})$ is not bounded below by $n - c$ for any constant c. Furthermore, Conjecture 5 implies that the error term for the first order approximation in Question 4 is at least $O(\log n)$.

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