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# Refining invariants for computing autotopism groups of partial Latin rectangles

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ABSTRACT

Prior to using computational tools that find the autotopism group of a partial Latin rectangle (its stabilizer group under row, column and symbol permutations), it is beneficial to find partitions of the rows, columns and symbols that are invariant under autotopisms and are as fine as possible. We look at the lattices formed by these partitions and introduce two invariant refining maps on these lattices. The first map generalizes the strong entry invariant in a previous work. The second map utilizes some bipartite graphs, introduced here, whose structure is determined by pairs of rows (or columns, or symbols). Experimental results indicate that in most cases (ordinarily 99%+), the combined use of these invariants gives the theoretical best partition of the rows, columns and symbols, outperforms the strong entry invariant, which only gives the theoretical best partitions in roughly 80% of the cases.



## 1. Introduction

Autotopisms of Latin squares (symmetries under row, column and symbol permutations), as well as the problem of computing the autotopism group of a Latin square, have been extensively studied during the last three decades and even earlier [2,5–8,10–13,16,27,31–34,41–43,47,48,53], with recent applications in cryptography, such as in secret sharing schemes [11,47] and graph coloring games [1,15]. Autotopisms of (partial) Latin rectangles, however, are less well-studied, and most research on the topic has been in the last decade [14,17–20].

Current methods for computing the autotopism group of a (partial) Latin square or rectangle are based either on backtracking, essentially an exhaustive search of worst-case super-polynomial-time complexity,[1] or on the computation of the automorphism group of a related graph. The now-standard way of computing the autotopism group $\mathrm{Atop}(L)$ of an $n \times n$ Latin square $L$ is by McKay, Meynert and Myrvold [38], which is achieved via a vertex-colored graph with automorphism group isomorphic to $\mathrm{Atop}(L)$: it has $n^2 + 3n$ vertices and $3n^2$ edges, and maximum degree $\max(3, n)$. The McKay, Meynert and Myrvold method readily generalizes to $r \times s$ partial Latin rectangles on $n$ symbols [19], in which case we have a vertex-colored bipartite graph with $m + r + s + n$ vertices and $3m$ edges, and maximum degree up to $\max(3, r, s, n)$.

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[1] A Latin square of order $n$ has at most $n^{O(\log n)}$ autotopisms [6] although such Latin squares are rare [40]; a $1 \times n$ Latin rectangle has $n!$ autotopisms. To compute a list of all autotopisms, it thus necessarily takes worst-case super-polynomial time.

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Through the construction in [45, Th. 4] (which embeds graphs into partial Latin squares), it is possible to solve the graph isomorphism (GI) problem for graphs $G$ and $H$ through constructing a partial Latin square with autotopism group isomorphic to the automorphism group of the graph $G \cup H$. Thus, computing the autotopism group of a partial Latin square requires at least polynomially equivalent time to solving GI. (This argument may also work for non-partial Latin squares via Phelps' construction [44], although proving this is less straightforward, since this would require accounting for the orders of the Latin squares involved.) The complexity of the GI problem was recently found to be at most $\exp((\log n)^{O(1)})$, where $n$ is the number of vertices, by Babai [3] (see also [4]).

To reduce the time spent on the computation of the automorphism groups of partial Latin square when using the aforementioned graph techniques, distinct autotopism invariants of (partial) Latin rectangles have been used [19,30,34,45], for which the complexity of their computation is polynomial in the dimensions of the array under consideration. These invariants partition the entries, rows, columns and/or symbols of the corresponding (partial) Latin rectangle so that the parts are preserved by autotopisms. The finer the partitions, the less time required to compute the autotopism group. Nevertheless, no polynomial-time-computable autotopism invariant is currently known so that the corresponding partitions are ensured to be the ones determined by the orbits of $\mathrm{Atop}(L)$, which are the theoretically finest partitions.

This paper introduces two methods of refining any given $\mathrm{Atop}(L)$-fixed partition of the rows, columns and symbols of a given partial Latin rectangle $L$. The first method generalizes the so-called strong entry invariants [19]. The second one is based on the isomorphism classes of some vertex-and-edge-colored bipartite graphs associated with every pair of rows, every pair of columns and every pair of symbols. These graphs generalize the cycle representation of permutations to partial permutations. These refinement methods arise from certain maps defined on the lattices of partitions of the rows, columns and symbols of a given partial Latin rectangle. We show that these maps are not lattice endomorphisms, although they possess some of the characteristics of such endomorphisms.

This paper is organized as follows. Section 2 describes preliminary concepts. In Section 3 we describe the natural refinement—a procedure to refine any given system of row, column and symbol partitions. In Section 4 we introduce a vertex-and-edge-colored bipartite graph related to any pair of rows, columns or symbols. In Section 5 we use the distribution of these graphs into isomorphism classes to define another partition refinement method. Section 6 contains some examples of particular interest. Section 7 gives experimental results comparing the effectiveness of different partitioning methods. We conclude in Section 8 with some ideas for future research. Appendix gives a glossary of repeatedly used notation.

## 2. Preliminaries

Let $[n] = \{1, \ldots, n\}$. An $r \times s$ *partial Latin rectangle* (or PLR for short) *based on* $[n]$ is an $r \times s$ array $L = (L[i, j])$ containing symbols from the set $[n] \cup \{\cdot\}$ such that each symbol in $[n]$ appears at most once in each row and each column. Throughout this paper, partial Latin rectangles are assumed to have (a) at least one entry in each row, (b) at least one entry in each column and (c) at least one copy of each symbol in $[n]$. Each partial Latin rectangle $L$ is uniquely determined by its *entry set*

$$\mathrm{Ent}(L) := \{(i, j, L[i, j]) \colon i \in [r], j \in [s], \text{ and } L[i, j] \in [n]\},$$

where rows and columns of $L$ are indexed in natural order by $[r]$ and $[s]$, respectively. Any triple $(i, j, L[i, j]) \in \mathrm{Ent}(L)$ is called an *entry* of $L$, and any pair $(i, j) \in [r] \times [s]$ is a *cell*. Any cell $(i, j)$ containing $\cdot$ is said to be *empty*. If $L$ does not have empty cells, then it is a *Latin rectangle*. A Latin rectangle with $r = s = n$ is a *Latin square of order n*. (In other papers, a "Latin rectangle" ordinarily requires $r \le s = n$, but here we do not use this restriction.) Let $\mathrm{PLR}(r, s, n)$ and $\mathrm{PLR}(r, s, n; m)$ denote the set of $r \times s$ partial Latin rectangles based on $[n]$ and its subset of rectangles with $m$ entries, respectively.

For an integer $t \ge 1$, let $S_t$ denote the symmetric group on the set $[t]$. Let $\pi \in S_3$ and $L \in \mathrm{PLR}(d_1, d_2, d_3)$. The *conjugate* partial Latin rectangle $L^\pi \in \mathrm{PLR}(d_{\pi(1)}, d_{\pi(2)}, d_{\pi(3)})$ is defined so that

$$\mathrm{Ent}(L^\pi) = \{(l_{\pi(1)}, l_{\pi(2)}, l_{\pi(3)}) \colon (l_1, l_2, l_3) \in \mathrm{Ent}(L)\}.$$

Thus, there exist six conjugates among which we emphasize $L$ itself, its transpose $L^{(12)}$, and the conjugate $L^{(13)}$, which results from switching rows and symbols.

The *isotopism group* $S_r \times S_s \times S_n$ acts on the sets $\mathrm{PLR}(r, s, n)$ and $\mathrm{PLR}(r, s, n; m)$, with each *isotopism* $\theta = (\alpha, \beta, \gamma) \in S_r \times S_s \times S_n$ permuting the rows, columns and symbols of any partial Latin rectangle $L = (L[i, j])$ by $\alpha$, $\beta$ and $\gamma$, respectively. We say that $L$ and the resulting $L^\theta$ are *isotopic*. Specifically,

$$\mathrm{Ent}(L^\theta) = \left\{\big(\alpha(i), \beta(j), \gamma(L[i, j])\big) \colon \big(i, j, L[i, j]\big) \in \mathrm{Ent}(L)\right\}.$$

If $L = L^\theta$, then $\theta$ is said to be an *autotopism* of $L$. The set $\mathrm{Atop}(L)$ of autotopisms of $L$ forms a group, called the *autotopism group* of $L$.

To speed up the computation to find the autotopism group, we utilize *autotopism invariants*: properties of the rows, columns, symbols and/or entries of partial Latin rectangles which only change under isotopisms that are not autotopisms of the Latin square. The following autotopism invariants have been considered in the literature.

- For partial Latin squares of order $n$, Keedwell [30] introduced the *row type*, *column type* and *symbol type* respectively as the number of entries per row, the number of entries per column, and the number of appearances of each symbol in $[n]$ within the partial Latin square under consideration. All these notions straightforwardly generalized to partial Latin rectangles in $\mathrm{PLR}(r, s, n)$. Since these quantities are preserved by autotopism, Stones [45] described them as *row*, *column* and *symbol invariants* and used them for computing autotopism groups of partial Latin squares.
- *Strong entry invariants* (or SEIs, for short) of $L = (L[i, j]) \in \mathrm{PLR}(r, s, n)$ were introduced in [19] as a triple for each entry $(i, j, L[i, j]) \in E(L)$, consisting of the row type of $i$, the column type of $j$ and the symbol type of $L[i, j]$. These invariants determine a natural relabeling of symbols within $L$ that gives rise to the *strong entry invariant matrix* [46]. The multisets of these relabeled symbols, corresponding to rows, columns and symbols in $L$ determine row, column and symbol invariants, respectively (see the next example).
- For Latin squares, Kotlar [33,34] introduced methods involving the cycle structure representation of permutations, which consists of an autotopism invariant attached to any pair of rows of the Latin square under consideration. The refinement method introduced in Section 4 and 5 generalizes the methods in [33] and [34] to partial Latin rectangles.

There are other invariants which have been sporadically used, such as the "train" invariant in [52], which was used for specific types of Latin squares. The graph-isomorphism software `Nauty` has in-built invariants, but these are designed for graphs in general, and are not specialized for partial Latin rectangles. We do not focus on these invariants in this paper.

Consider the partial Latin square

$$L = \begin{array}{|c|c|c|c|c|c|}\hline 4 & 3 & 2 & 1 & 5 & \cdot \\ \hline 1 & 4 & 5 & 6 & 3 & 2 \\ \hline 2 & \cdot & 1 & 3 & 4 & 6 \\ \hline 5 & 2 & 3 & 4 & 6 & 1 \\ \hline 3 & 1 & 6 & 5 & 2 & 4 \\ \hline 6 & 5 & \cdot & 2 & 1 & 3 \\ \hline \end{array}. \tag{2.1}$$

After replacing each symbol in $L$ by the corresponding SEI, we get

$$\begin{array}{|c|c|c|c|c|c|}\hline (5,6,5) & (5,5,6) & (5,5,6) & (5,6,6) & (5,6,5) & \cdot \\ \hline (6,6,6) & (6,5,5) & (6,5,5) & (6,6,5) & (6,6,6) & (6,5,6) \\ \hline (5,6,6) & \cdot & (5,5,6) & (5,6,6) & (5,6,5) & (5,5,5) \\ \hline (6,6,5) & (6,5,6) & (6,5,6) & (6,6,5) & (6,6,5) & (6,5,6) \\ \hline (6,6,6) & (6,5,6) & (6,5,5) & (6,6,5) & (6,6,6) & (6,5,5) \\ \hline (5,6,5) & (5,5,5) & \cdot & (5,6,6) & (5,6,6) & (5,5,6) \\ \hline \end{array}.$$

Here, for example, the SEI $(5, 6, 5)$ for the entry $(1, 1, 4) \in \mathrm{Ent}(L)$ signifies that $L$ has five entries in row 1, six entries in column 1 and the symbol 4 appears five times within $L$. The strong entry invariant matrix of $L$ is then obtained by relabeling each SEI by a positive integer so that (a) distinct integers appear in natural order when read row by row then column by column, and (b) two SEIs are equal if and only if they are equally labeled. Thus, the strong entry invariant matrix corresponding to $L$ is

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & 1 & \cdot \\ \hline 4 & 5 & 5 & 6 & 4 & 7 \\ \hline 3 & \cdot & 2 & 3 & 1 & 8 \\ \hline 6 & 7 & 7 & 6 & 6 & 7 \\ \hline 4 & 7 & 5 & 6 & 4 & 5 \\ \hline 1 & 8 & \cdot & 3 & 3 & 2 \\ \hline \end{array}. \tag{2.2}$$

As examples of multisets derived from these SEIs, the multiset associated with row 1 of $L$ is $\{1, 1, 2, 2, 3\}$, for column 3 is $\{2, 2, 5, 5, 7\}$, and for the cells containing the symbol 2 in $L$ the multiset is $\{2, 3, 3, 4, 7, 7\}$.

By partitioning the rows, columns and symbols based on equality of the multisets associated to each row, column and symbol, we obtain the three partitions:

$$\begin{aligned} \text{row partition: } &\{\{1\}, \{2, 5\}, \{3, 6\}, \{4\}\}, \\ \text{column partition: } &\{\{1, 5\}, \{2, 6\}, \{3\}, \{4\}\}, \text{ and} \\ \text{symbol partition: } &\{\{1, 2\}, \{3\}, \{4\}, \{5, 6\}\}. \end{aligned}$$

Several methods for computing autotopism groups of partial Latin rectangles, with and without SEIs, were experimentally compared by four of the present authors in [46]. We use SEIs as a baseline for our experiments. It was observed that the SEIs are less effective for very dense and very sparse partial Latin rectangles, as sparse and dense partial Latin rectangles have less variety of SEIs. Moreover, SEIs are useless for (a) non-partial Latin rectangles with $s = n$, where each strong entry invariant is $(r, n, n)$, and (b) $k$-plexes [51] (also known as $k$-protoplexes), which are partial Latin squares with $k$ entries in each row and column, and $k$ copies of each symbol, where each strong entry invariant is $(k, k, k)$. These limitations motivate further study on new invariants.

## 3. The natural refinement

In general, a finer partition of the rows, columns and symbols resulting from an invariant (such as in the previous section) implies less subsequent run-time for computing $\mathrm{Atop}(L)$. This section describes a method for refining such given partitions.

We define a *system of partitions* on $L \in \mathrm{PLR}(r, s, n)$ as a triple

$$\mathscr{P} = (\mathcal{P}_{\mathrm{row}}, \mathcal{P}_{\mathrm{col}}, \mathcal{P}_{\mathrm{sym}})$$

where $\mathcal{P}_{\mathrm{row}}$ is a partition of the rows, $\mathcal{P}_{\mathrm{col}}$ a partition of the columns and $\mathcal{P}_{\mathrm{sym}}$ a partition of the symbols of $L$. We investigate systems of partitions which are fixed under the action of autotopisms. More specifically, $\mathscr{P}$ is *fixed* (setwise) under $\mathrm{Atop}(L)$ if for any autotopism $(\alpha, \beta, \gamma) \in \mathrm{Atop}(L)$, the partition $\mathcal{P}_{\mathrm{row}}$ is fixed by $\alpha$, the partition $\mathcal{P}_{\mathrm{col}}$ is fixed by $\beta$, and the partition $\mathcal{P}_{\mathrm{sym}}$ is fixed by $\gamma$. From here on, we denote respectively $\mathrm{SPart}(L)$ and $\mathrm{SPart}_{\mathrm{Atop}}(L)$ as the set of all systems of partitions on $L$ and its subset of systems of partitions on $L$ that are fixed under $\mathrm{Atop}(L)$. We aim at finding the finest system of partitions within the set $\mathrm{SPart}_{\mathrm{Atop}}(L)$, which we call the *orbit-system of partitions* $\mathscr{P}_{\mathrm{O}}(L)$ of $L$. In $\mathscr{P}_{\mathrm{O}}(L)$, the partitions of the rows, the columns and the symbols are defined by orbits under the action of $\mathrm{Atop}(L)$ (for example, two rows $i$ and $i'$ belong to the same part in the first component of $\mathscr{P}_{\mathrm{O}}(L)$ whenever there is an autotopism which maps row $i$ to row $i'$). For brevity, we also refer to the orbit-system of partitions as the *orbits system*.

A system of partitions $\mathscr{P}$ on $L$ is fixed under the action of $\mathrm{Atop}(L)$ if and only if the orbit-system of partitions $\mathscr{P}_{\mathrm{O}}(L)$ is finer than $\mathscr{P}$. Thus, $\mathscr{P}_{\mathrm{O}}(L)$ is the theoretical best (the finest) partition of the rows, columns, and symbols of $L$ we can compute, and we set this as the goal in the subsequent experiments.

For a partial Latin rectangle $L$ we use $\mathscr{P}_{\mathrm{T}}(L)$ to denote the system of partitions arising from $L$'s row, column and symbol types, and we let $\mathscr{P}_{\mathrm{SEI}}(L)$ denote the system of partitions on $L$ arising from its SEIs (see Section 2). The systems $\mathscr{P}_{\mathrm{T}}(L)$ and $\mathscr{P}_{\mathrm{SEI}}(L)$ are examples of systems that are fixed under $\mathrm{Atop}(L)$.

In general, the set $\mathrm{Part}(A)$ of all partitions of a given set $A$ may be endowed with a partial order $\leq$ such that two partitions $\mathcal{P}, \mathcal{P}' \in \mathrm{Part}(A)$ hold that $\mathcal{P} \leq \mathcal{P}'$ if and only if $\mathcal{P}$ is finer than (or equal to) $\mathcal{P}'$. As such, the triple $(\mathrm{Part}(A), \wedge, \vee)$ is a complete lattice [22, Chapter IV, §4, Lemma 1], which is called the *partition lattice* of $A$. Here, $\wedge$ and $\vee$ are two binary operation on the set $\mathrm{Part}(A)$ such that, for each pair of partitions $\mathcal{P}, \mathcal{P}' \in \mathrm{Part}(A)$,

- $\mathcal{P} \wedge \mathcal{P}'$ is the coarsest among all the partitions in $\mathrm{Part}(A)$ refining both $\mathcal{P}$ and $\mathcal{P}'$, and
- $\mathcal{P} \vee \mathcal{P}'$ is the finest among all the systems of partitions in $\mathrm{Part}(A)$ that are coarser than both $\mathcal{P}$ and $\mathcal{P}'$.

From here on, we write $a \sim_{\mathcal{P}} b$ if $a$ and $b$ belong to the same part in a given partition $\mathcal{P}$. The following result characterizes both operations $\wedge$ and $\vee$ by describing relations on elements.

**Lemma 1** (*[22, Chapter IV, §4, Lemma 1]*)**.** *Let* $(\mathrm{Part}(A), \wedge, \vee)$ *be the partition lattice of a set A. Let* $\mathcal{P}, \mathcal{P}' \in \mathrm{Part}(A)$ *and* $a, b \in A$. *Then,*

*(i)* $a \sim_{\mathcal{P}\wedge\mathcal{P}'} b$ *if and only if* $a \sim_{\mathcal{P}} b$ *and* $a \sim_{\mathcal{P}'} b$.
*(ii)* $a \sim_{\mathcal{P}\vee\mathcal{P}'} b$ *if and only if there exists a sequence of elements* $c_0, \ldots, c_{k+1} \in A$ *and a sequence of partitions* $\mathcal{P}_0, \ldots, \mathcal{P}_k \in \{\mathcal{P}, \mathcal{P}'\}$ *such that* $a = c_0$, $b = c_{k+1}$ *and* $c_i \sim_{\mathcal{P}_i} c_{i+1}$, *for all* $i \in \{0, \ldots, k\}$.

In a similar way, a componentwise partial order may be described in a natural way so that both sets $\mathrm{SPart}(L)$ and $\mathrm{SPart}_{\mathrm{Atop}}(L)$ are endowed with a complete lattice structure, for any given partial Latin rectangle $L \in \mathrm{PLR}(r, s, n)$. In this regard, we denote $\mathscr{P} \leq \mathscr{P}'$ when a system of partitions $\mathscr{P}$ is componentwise finer than a system of partitions $\mathscr{P}'$. Moreover, both sets $\mathrm{SPart}(L)$ and $\mathrm{SPart}_{\mathrm{Atop}}(L)$ are bounded lattices. Their common maximum (the coarsest partition) is the 1-*part partition* or *trivial partition*,

$$\mathscr{P}_1(L) := (\{[r]\}, \{[s]\}, \{[n]\}).$$

The minimum (the finest partition) of the set $\mathrm{SPart}(L)$ is the *partition of singletons*,

$$\mathscr{P}_{\mathrm{S}}(L) := (\{\{1\}, \ldots, \{r\}\}, \{\{1\}, \ldots, \{s\}\}, \{\{1\}, \ldots, \{n\}\}),$$

whereas the minimum of $\mathrm{SPart}_{\mathrm{Atop}}(L)$ is the orbit-system of partitions $\mathscr{P}_{\mathrm{O}}(L)$. Notice in this regard that, unlike $\mathscr{P}_1(L)$, which is always fixed under $\mathrm{Atop}(L)$, the partition of singletons $\mathscr{P}_{\mathrm{S}}(L)$ is fixed under $\mathrm{Atop}(L)$ if and only if $\mathrm{Atop}(L)$ is trivial.

In the context of a partial Latin rectangle $L \in \mathrm{PLR}(r, s, n)$, partitions of $[r]$ are called *row partitions*, and we likewise define *column partitions*, *symbol partitions*, and *entry partitions* of $[s]$, $[n]$ and $\mathrm{Ent}(L)$, respectively. Let us define the map

$$E : \mathrm{SPart}(L) \to \mathrm{Part}(\mathrm{Ent}(L))$$

so that, for any $\mathscr{P} = (\mathcal{P}_{\mathrm{row}}, \mathcal{P}_{\mathrm{col}}, \mathcal{P}_{\mathrm{sym}}) \in \mathrm{SPart}(L)$, two entries $(i, j, L[i, j])$ and $(i', j', L[i', j'])$ belong to the same part in the entry partition $E(\mathscr{P}) \in \mathrm{Part}(\mathrm{Ent}(L))$ if and only if $i \sim_{\mathcal{P}_{\mathrm{row}}} i'$, and $j \sim_{\mathcal{P}_{\mathrm{col}}} j'$, and $L[i, j] \sim_{\mathcal{P}_{\mathrm{sym}}} L[i', j']$. (While $E(\mathscr{P})$ varies with $L$, we omit $L$ from this notation.)

**Proposition 1.** *Let L be any given partial Latin rectangle. The map* $E : \mathrm{SPart}(L) \to \mathrm{Part}(\mathrm{Ent}(L))$ *has the following properties:*

*(i) It is order-preserving. That is, for every $\mathscr{P}, \mathscr{P}' \in \mathrm{SPart}(L)$, if $\mathscr{P}' \leq \mathscr{P}$, then $E(\mathscr{P}') \leq E(\mathscr{P})$.*

*(ii) It is a lattice homomorphism. That is, for every $\mathscr{P}, \mathscr{P}' \in \mathrm{SPart}(L)$,*

*(a) $E(\mathscr{P} \wedge \mathscr{P}') = E(\mathscr{P}) \wedge E(\mathscr{P}')$. (That is, $E$ is a meet-homomorphism.)*

*(b) $E(\mathscr{P} \vee \mathscr{P}') = E(\mathscr{P}) \vee E(\mathscr{P}')$. (That is, $E$ is a join-homomorphism.)*

**Proof.** Let $\mathscr{P} = (\mathcal{P}_{\mathrm{row}}, \mathcal{P}_{\mathrm{col}}, \mathcal{P}_{\mathrm{sym}})$ and $\mathscr{P}' = (\mathcal{P}'_{\mathrm{row}}, \mathcal{P}'_{\mathrm{col}}, \mathcal{P}'_{\mathrm{sym}})$ be two systems of partitions in $\mathrm{SPart}(L)$, and let $e = (i, j, L[i,j])$ and $e' = (i', j', L[i', j'])$ be two entries in $\mathrm{Ent}(L)$. We prove each assertion separately.

(i) If $e \sim_{E(\mathscr{P}')} e'$, then, by definition, $i \sim_{\mathcal{P}'_{\mathrm{row}}} i'$, and $j \sim_{\mathcal{P}'_{\mathrm{col}}} j'$ and $L[i,j] \sim_{\mathcal{P}'_{\mathrm{sym}}} L[i',j']$. If $\mathscr{P}' \leq \mathscr{P}$, then we have that $i \sim_{\mathcal{P}_{\mathrm{row}}} i'$, and $j \sim_{\mathcal{P}_{\mathrm{col}}} j'$, and $L[i,j] \sim_{\mathcal{P}_{\mathrm{sym}}} L[i',j']$. Thus, $e \sim_{E(\mathscr{P})} e'$.

(ii) We have from (i) that

(a) $E(\mathscr{P} \wedge \mathscr{P}') \leq E(\mathscr{P})$ and $E(\mathscr{P} \wedge \mathscr{P}') \leq E(\mathscr{P}')$. Since $E(\mathscr{P}) \wedge E(\mathscr{P}')$ is the coarsest among all the partitions refining both $E(\mathscr{P})$ and $E(\mathscr{P}')$ we have that

$$E(\mathscr{P} \wedge \mathscr{P}') \leq E(\mathscr{P}) \wedge E(\mathscr{P}').$$

Now, if $e \sim_{E(\mathscr{P}) \wedge E(\mathscr{P}')} e'$, then Lemma 1(i) implies that $e \sim_{E(\mathscr{P})} e'$ and $e \sim_{E(\mathscr{P}')} e'$. So, $i \sim_{\mathcal{P}_{\mathrm{row}}} i'$, and $j \sim_{\mathcal{P}_{\mathrm{col}}} j'$, and $L[i,j] \sim_{\mathcal{P}_{\mathrm{sym}}} L[i',j']$, and $i \sim_{\mathcal{P}'_{\mathrm{row}}} i'$, and $j \sim_{\mathcal{P}'_{\mathrm{col}}} j'$, and $L[i,j] \sim_{\mathcal{P}'_{\mathrm{sym}}} L[i',j']$. Again, from Lemma 1(i), $i \sim_{\mathcal{P}_{\mathrm{row}} \wedge \mathcal{P}'_{\mathrm{row}}} i'$, and $j \sim_{\mathcal{P}_{\mathrm{col}} \wedge \mathcal{P}'_{\mathrm{col}}} j'$, and $L[i,j] \sim_{\mathcal{P}_{\mathrm{sym}} \wedge \mathcal{P}'_{\mathrm{sym}}} L[i',j']$. Thus, by the definition of the map $E$, $e \sim_{E(\mathscr{P} \wedge \mathscr{P}')} e'$, and hence,

$$E(\mathscr{P}) \wedge E(\mathscr{P}') \leq E(\mathscr{P} \wedge \mathscr{P}').$$

(b) $E(\mathscr{P}) \leq E(\mathscr{P} \vee \mathscr{P}')$ and $E(\mathscr{P}') \leq E(\mathscr{P} \vee \mathscr{P}')$. Since $E(\mathscr{P}) \vee E(\mathscr{P}')$ is the finest among all the partitions refined by both $E(\mathscr{P})$ and $E(\mathscr{P}')$ we have that

$$E(\mathscr{P}) \vee E(\mathscr{P}') \leq E(\mathscr{P} \vee \mathscr{P}').$$

Now, if $e \sim_{E(\mathscr{P} \vee \mathscr{P}')} e'$, then $i \sim_{\mathcal{P}_{\mathrm{row}} \vee \mathcal{P}'_{\mathrm{row}}} i'$, and $j \sim_{\mathcal{P}_{\mathrm{col}} \vee \mathcal{P}'_{\mathrm{col}}} j'$, and $L[i,j] \sim_{\mathcal{P}_{\mathrm{sym}} \vee \mathcal{P}'_{\mathrm{sym}}} L[i',j']$. From Lemma 1(ii), the following sequences exist:

(i) $r_0, \ldots, r_{k+1} \in [r]$ and $\mathcal{P}^r_0, \ldots, \mathcal{P}^r_k \in \{\mathcal{P}_{\mathrm{row}}, \mathcal{P}'_{\mathrm{row}}\}$ such that $i = r_0$, $i' = r_{k+1}$ and $r_i \sim_{\mathcal{P}^r_i} r_{i+1}$, for all $i \in \{0, \ldots, k\}$.

(ii) $c_0, \ldots, c_{m+1} \in [s]$ and $\mathcal{P}^c_0, \ldots, \mathcal{P}^c_m \in \{\mathcal{P}_{\mathrm{col}}, \mathcal{P}'_{\mathrm{col}}\}$ such that $j = c_0$, $j' = c_{m+1}$ and $c_i \sim_{\mathcal{P}^c_i} c_{i+1}$, for all $i \in \{0, \ldots, m\}$.

(iii) $s_0, \ldots, s_{\ell+1} \in [n]$ and $\mathcal{P}^s_0, \ldots, \mathcal{P}^s_\ell \in \{\mathcal{P}_{\mathrm{sym}}, \mathcal{P}'_{\mathrm{sym}}\}$ such that $L[i,j] = s_0$, $L[i',j'] = s_{\ell+1}$ and $s_i \sim_{\mathcal{P}^s_i} s_{i+1}$, for all $i \in \{0, \ldots, \ell\}$.

Then, we define the sequence $t_0, \ldots, t_{k+m+\ell} \in [r] \times [s] \times [n]$ so that

$$t_i = \begin{cases} (r_i, c_0, s_0), & \text{for all } i \in \{0, \ldots, k\}, \\ (i', c_{i-k}, s_0), & \text{for all } i \in \{k+1, \ldots, k+m\}, \\ (i', j', s_{i-k-m}), & \text{for all } i \in \{k+m+1, \ldots, k+m+\ell\}. \end{cases}$$

Thus, $e = t_0$, $e' = t_{k+m+\ell}$ and, for any $i \in \{0, \ldots, k+m+\ell-1\}$, and so $t_i \sim_{E(\mathscr{P})} t_{i+1}$ or $t_i \sim_{E(\mathscr{P}')} t_{i+1}$. Then, from Lemma 1(ii), $(i, j, L[i,j]) \sim_{E(\mathscr{P}) \vee E(\mathscr{P}')} (i', j', L[i',j'])$, and hence,

$$E(\mathscr{P} \vee \mathscr{P}') \leq E(\mathscr{P}) \vee E(\mathscr{P}'). \quad \square$$

Now, let us describe how the lattice homomorphism $E$ may be used to refine any system of partitions of a partial Latin rectangle $L \in \mathrm{PLR}(r, s, n)$. To this end, we introduce a new map

$$F : \mathrm{Part}(\mathrm{Ent}(L)) \to \mathrm{SPart}(L)$$

so that, for any partition $\mathcal{P} \in \mathrm{Part}(\mathrm{Ent}(L))$, the new system of partition $F(\mathcal{P}) := (F_{\mathrm{row}}(\mathcal{P}), F_{\mathrm{col}}(\mathcal{P}), F_{\mathrm{sym}}(\mathcal{P}))$ is defined as follows. Firstly, we label the entries of $L$ with $\{1, 2, \ldots\}$ in such a way that two entries have the same label if and only if they belong to the same part of $\mathcal{P}$. Then, we say that two rows are in the same part of $F_{\mathrm{row}}(\mathcal{P})$ if and only if the multisets of labels of the entries in those rows are equal. The partitions $F_{\mathrm{col}}(\mathcal{P})$ and $F_{\mathrm{sym}}(\mathcal{P})$ are defined analogously. (Here also, $F(\mathcal{P})$ varies with $L$, but we omit $L$ from this notation.)

**Lemma 2.** *Let $L$ be a partial Latin rectangle. The map $F : \mathrm{Part}(\mathrm{Ent}(L)) \to \mathrm{SPart}(L)$ satisfies the following properties:*

*(i) $F$ is order-preserving.*

*(ii) For each $\mathcal{P}, \mathcal{P}' \in \mathrm{Part}(\mathrm{Ent}(L))$,*

*(a) $F(\mathcal{P} \wedge \mathcal{P}') \leq F(\mathcal{P}) \wedge F(\mathcal{P}')$; and*

*(b) $F(\mathcal{P}) \vee F(\mathcal{P}') \leq F(\mathcal{P} \vee \mathcal{P}')$.*

**Proof.** The second statement follows readily from the first. In order to prove (i), let $\mathcal{P}, \mathcal{P}' \in \text{Part}(\text{Ent}(L))$ be such that $\mathcal{P}' \leq \mathcal{P}$ and let us prove that $F_{\text{row}}(\mathcal{P}') \leq F_{\text{row}}(\mathcal{P})$. (The proofs for $F_{\text{col}}$ and $F_{\text{sym}}$ follow analogously.) Let $i, j \in [r]$ be such that $i \sim_{F_{\text{row}}(\mathcal{P}')} j$. Thus, rows $i$ and $j$ have the same multisets of labels arising from $\mathcal{P}'$. Since $\mathcal{P}' \leq \mathcal{P}$, rows $i$ and $j$ also have the same multisets of labels arising from $\mathcal{P}$. Thus, $i \sim_{F_{\text{row}}(\mathcal{P})} j$. □

In order to illustrate that $F$ is not a meet-homomorphism, let us consider the following two entry partitions in matrix form of a given Latin square of order three.

$$\mathcal{P}_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 2 & 1 \\ \hline 3 & 4 & 3 \\ \hline\end{array} \qquad \mathcal{P}_2 = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 1 & 1 & 2 \\ \hline 1 & 1 & 2 \\ \hline\end{array}$$

Then, the matrix form of the entry partition $\mathcal{P}_1 \wedge \mathcal{P}_2$ is

$$\mathcal{P}_1 \wedge \mathcal{P}_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 2 & 4 \\ \hline 5 & 6 & 7 \\ \hline\end{array}$$

Hence,

$$F_{\text{row}}(\mathcal{P}_1 \wedge \mathcal{P}_2) = \{\{1\}, \{2\}, \{3\}\} < \{\{1, 2\}, \{3\}\} = F_{\text{row}}(\mathcal{P}_1) \wedge F_{\text{row}}(\mathcal{P}_2).$$

Further, in order to illustrate that $F$ is not a join-homomorphism, let us consider the following two entry partitions in matrix form of any given Latin square of order three.

$$\mathcal{P} = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 3 & 4 \\ \hline 5 & 1 & 4 \\ \hline\end{array} \qquad \mathcal{P}' = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 2 & 5 \\ \hline 4 & 6 & 6 \\ \hline\end{array}$$

Then, the matrix form of the entry partition $\mathcal{P} \vee \mathcal{P}'$ is

$$\mathcal{P} \vee \mathcal{P}' = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 2 & 1 \\ \hline 2 & 1 & 1 \\ \hline\end{array}$$

Hence,

$$F_{\text{row}}(\mathcal{P}) \vee F_{\text{row}}(\mathcal{P}') = \{\{1\}, \{2\}, \{3\}\} < \{\{1, 2\}, \{3\}\} = F_{\text{row}}(\mathcal{P} \vee \mathcal{P}').$$

The composition of the lattice homomorphism $E$ with the order-preserving map $F$ enables us to introduce the *natural refinement* of a partial Latin rectangle $L$ as the map

$$N : \text{SPart}(L) \to \text{SPart}(L)$$

so that $N = F \circ E$. It is analogous to the "refinement" in the individualization-refinement process used by Nauty [37,39]. As the name suggests, we use it to refine systems of partitions, specifically those that we know are fixed under $\text{Atop}(L)$, such as those deduced through invariants, as per the following theorem.

**Theorem 1.** *Let $L$ be a PLR. The natural refinement $N : \text{SPart}(L) \to \text{SPart}(L)$ satisfies the following properties:*

*(i) $N$ is a refinement;*
*(ii) $N$ maps $\text{SPart}_{\text{Atop}}(L)$ to itself; and*
*(iii) $N$ is order-preserving*

**Proof.** Let $\mathscr{P} = (\mathcal{P}_{\text{row}}, \mathcal{P}_{\text{col}}, \mathcal{P}_{\text{sym}}) \in \text{SPart}(L)$ and $N(\mathscr{P}) = (N_{\text{row}}(\mathscr{P}), N_{\text{col}}(\mathscr{P}), N_{\text{sym}}(\mathscr{P})) \in \text{SPart}(L)$. We prove each statement for $N_{\text{row}}(\mathscr{P})$. The proofs for $N_{\text{col}}(\mathscr{P})$ and $N_{\text{sym}}(\mathscr{P})$ follow analogously.

(i) By definition, two rows that belong to the same part in $N_{\text{row}}(\mathscr{P})$ have the same multisets of labels, and thus belong to the same part in $\mathcal{P}_{\text{row}}$ (by the definition of $E(\mathscr{P})$). Thus, $N_{\text{row}}(\mathscr{P}) \leq \mathcal{P}_{\text{row}}$.
(ii) To prove the contrapositive: if an autotopism maps row $i$ to row $\alpha(i)$ and they belong to distinct parts of $N_{\text{row}}(\mathscr{P})$, then the two rows have distinct multisets of labels, so the autotopism maps some entry to another entry with a distinct label, and thus (by definition) these two entries belong to distinct parts of $E(\mathscr{P})$, which implies that $\mathscr{P}$ is not fixed under $\text{Atop}(L)$.
(iii) Follows readily from the fact that both $E$ and $F$ are order preserving. □

Let us remark that the map $N$ is not a lattice endomorphism. To illustrate this let us consider the Latin square

$$L = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline\end{array}$$

and the two systems of partitions in $\mathrm{SPart}(L)$

$$\mathscr{P}_1 = (\{\{1,2\},\{3\}\},\{\{1,2,3\}\},\{\{1\},\{2,3\}\})$$

and

$$\mathscr{P}_2 = (\{\{1,2,3\}\},\{\{1,2\},\{3\}\},\{\{1,2,3\}\}).$$

Notice that the matrix forms of the entry partitions $E(\mathscr{P}_1)$ and $E(\mathscr{P}_2)$ in $\mathrm{Ent}(L)$ coincide with the arrays $\mathcal{P}_1$ and $\mathcal{P}_2$ that were described just after Lemma 2. As a consequence, the row partitions of $N(\mathcal{P}_1 \wedge \mathcal{P}_2)$ and $N(\mathcal{P}_1) \wedge N(\mathcal{P}_2)$ do not coincide and hence, $N$ is not even a meet-homomorphism.

The following proposition describes the natural refinement of certain systems of partitions. We remind the reader that notation is tabulated in the Appendix.

**Proposition 2.** *Let $L \in \mathrm{PLR}(r,s,n)$. The system of 1-part partitions $\mathscr{P}_1(L)$ satisfies*

$$N(\mathscr{P}_1(L)) = \mathscr{P}_{\mathrm{T}}(L)$$

*and the system of types $\mathscr{P}_{\mathrm{T}}(L)$ satisfies*

$$N(\mathscr{P}_{\mathrm{T}}(L)) = \mathscr{P}_{\mathrm{SEI}}(L).$$

*Further, the systems of partitions $\mathscr{P}_{\mathrm{T}}(L)$ and $\mathscr{P}_{\mathrm{SEI}}(L)$ constitute the respective maxima of the sets $\{N(\mathcal{P})\colon \mathcal{P} \in \mathrm{SPart}(L)\}$ and $\{N^2(\mathcal{P})\colon \mathcal{P} \in \mathrm{SPart}(L)\}$.*

**Proof.** We prove this for row partitions; the arguments for the column and symbol partitions are similar. The partition $E(\mathscr{P}_1(L))$ is a 1-part partition, so all the entries of $L$ are ascribed the same label in the construction of the natural refinement $N(\mathscr{P}_1(L))$. Thus, the partition of the rows in this refinement only depends on the sizes of the multisets of labels in each row, which implies the first statement. The second statement follows directly from the definition of $\mathscr{P}_{\mathrm{SEI}}(L)$. Finally, note that if two rows in $L$ are ascribed the same multisets of labels in the construction of the natural refinement $N(\mathscr{P})$ then, in particular, both multisets have the same size, and thus, both rows have the same type. $\square$

Let $\mathscr{P} \in \mathrm{SPart}(L)$ for a partial Latin rectangle $L$. We apply $N$ repeatedly:

$$\cdots \leq N^k(\mathscr{P}) \leq \ldots \leq N^2(\mathscr{P}) \leq N(\mathscr{P}) \leq \mathscr{P}, \tag{3.1}$$

until no more refinement is achieved. We call the finest system of partitions in the chain the *exhaustive natural refinement* of $\mathscr{P}$, which we denote $N^\infty(\mathscr{P})$. The exhaustive natural refinement is well-defined by Theorem 1(i) (and since we are working with finite partial Latin rectangles).

Notice that the family $\{N^\infty(\mathscr{P})\colon \mathscr{P} \in \mathrm{SPart}(L)\}$ constitutes the set of fixed points of the order-preserving map $N$ in the complete lattice $\mathrm{SPart}(L)$. Then, Tarski's fixed point theorem [49] implies that such a family also forms a complete lattice. The same holds for the family $\{N^\infty(\mathscr{P})\colon \mathscr{P} \in \mathrm{SPart}_{\mathrm{Atop}}(L)\}$. The maxima and minima of both complete lattices coincide with those of $\mathrm{SPart}(L)$ and $\mathrm{SPart}_{\mathrm{Atop}}(L)$, which were indicated after Lemma 1.

For the example of partial Latin rectangle $L$ given by (2.1) in Section 2, if we choose $\mathscr{P} = \mathscr{P}_{\mathrm{T}}(L)$, then the entry partition $E(\mathscr{P})$ in matrix form is given by (2.2). By Proposition 2, $N(\mathscr{P}) = \mathscr{P}_{\mathrm{SEI}}(L)$ and its entry partition $E(\mathscr{P}_{\mathrm{SEI}}(L))$ in matrix form is

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | · |
| 6 | 7 | 8 | 9 | 10 | 11 |
| 12 | · | 13 | 14 | 15 | 16 |
| 17 | 18 | 19 | 20 | 17 | 18 |
| 10 | 11 | 8 | 9 | 6 | 7 |
| 21 | 16 | · | 22 | 12 | 23 |

,

which gives rise to the system of partitions $N^2(\mathscr{P})$ whose components are

row partition: $\{\{1\},\{2,5\},\{3\},\{4\},\{6\}\}$,
column partition: $\{\{1\},\{2\},\ldots,\{6\}\}$, and
symbol partition: $\{\{1\},\{2\},\ldots,\{6\}\}$.

Thus, $\mathrm{Atop}(L)$ is trivial.

We analyze the complexity of the natural refinement. Throughout this paper, we employ the RAM model for analyzing complexity. Under this complexity model, memory reads and writes and integer comparisons all require 1 time step (regardless of the length of the integer).

Partial Latin rectangles can be stored in two formats: as a vector of entries, or as a matrix of symbols. If $M = \max(r,s,n)$, then converting from one format to the other takes $O(M^2)$ time, since there are at most $rs \leq M^2$ entries.

We assume the vector-of-entries format is stored sorted first by rows, then by columns, as would be normal in practice. If not, converting it to a matrix, then back again sorts it in $O(M^2)$ time, which does not affect the complexity results in this paper.

**Theorem 2.** *Let $\mathscr{P} = (\mathcal{P}_{\text{row}}, \mathcal{P}_{\text{col}}, \mathcal{P}_{\text{sym}})$ be a system of partitions on $L \in \mathrm{PLR}(r, s, n; m)$ and let $M = \max(r, s, n)$. The complexity of computing the natural refinement $N(\mathscr{P})$ is at most $O(M^2 \log M)$ and the complexity of computing the exhaustive natural refinement $N^\infty(\mathscr{P})$ is at most $O(M^3 \log M)$.*

**Proof.** We compute $N^\infty(\mathscr{P})$ by recursively computing $N^i(\mathscr{P})$. We stop when we encounter $N^k(\mathscr{P}) = N^{k-1}(\mathscr{P})$ for some $k$, which happens after at most $O(r+s+n) = O(M)$ iterations (since after each iteration, we strictly refine at least one of the partitions). It is thus sufficient to show that computing $N^{i+1}(\mathscr{P})$ from $N^i(\mathscr{P})$ has complexity $O(M^2 \log M)$.

To compute $N^{i+1}(\mathscr{P})$, we first compute the entry partition $E(N^i(\mathscr{P}))$. To this end, each of the $O(M^2)$ entries is ascribed an initial label which describes which parts of $N^i(\mathscr{P})$ its row, column and symbol belong to; for each entry this has complexity $O(1)$. We then relabel each entry using $\{1, 2, \ldots\}$ as follows: proceeding entry by entry, we either assign a new integer label (if the initial label has not been seen before) or reuse an integer label (if the initial label has been seen before). By maintaining a sparse array which stores the map between initial labels and integer labels, we can compute this integer relabeling in time $O(M^2)$. Thus, computing a matrix containing the values of $E(N^i(\mathscr{P}))$ has complexity $O(M^2)$.

Next, from $E(N^i(\mathscr{P}))$, we compute the multisets of labels in each row, in each column and for each symbol. To this end, we enumerate the integer labels in each "line" (the set of entries with some specific row, column or symbol index), generating a length-$O(M)$ vector of frequencies. Generating all such vectors has complexity $O(M^2)$.

We next identify the partitions (of the rows, columns and symbols) determined by the equality of these multisets, which we achieve through sorting according to the vectors of frequencies. Sorting uses $O(M \log M)$ comparisons of the $O(M)$ integers in the vector of frequencies. □

## 4. Two-line graphs

In this section, we describe a family of bipartite graphs that give rise to another partition refinement.

**Definition 1.** Let $L \in \mathrm{PLR}(r, s, n)$ and let $i, j \in [r]$ be two distinct rows of $L$. We define the $(i, j)$-*row graph* $G^{\text{row}}_{ij}(L)$ of $L$ as the following vertex- and edge-2-colored bipartite graph:

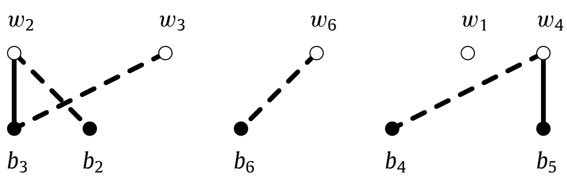
- The first part consists of white vertices labeled $w_u$, corresponding to the entries in row $i$, where $\{u\}$ are the corresponding symbols.
- The second part consists of black vertices labeled $b_v$, corresponding to the entries in row $j$, where $\{v\}$ are the corresponding symbols.
- Solid edges connect two vertices corresponding to entries in the same column.
- Dashed edges connect two vertices corresponding to entries with the same symbol.

For convenience, we place a white vertex above a black vertex if and only if they correspond to entries in the same column. So, solid edges are always vertical and dashed vertices are always diagonal.

For example, for the partial Latin rectangle

$$L = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 2 & \cdot & 3 & \cdot & 6 & \cdot & 1 & 4 \\ \hline 3 & 2 & \cdot & 6 & \cdot & 4 & \cdot & 5 \\ \hline \end{array} \tag{4.1}$$

the $(1, 2)$-row graph $G^{\text{row}}_{12}(L)$ is




We use the following vertex labels: vertices $(i, k, L[i, k])$ are relabeled $w_{L[i,k]}$, and vertices $(j, k, L[j, k])$ are relabeled $b_{L[j,k]}$.

For distinct columns $i, j \in [s]$, we also define the $(i, j)$-*column graph* $G^{\text{col}}_{ij}(L)$ as the $(i, j)$-row graph of its transpose $L^{(12)}$. Similarly, for distinct symbols $i, j \in [n]$, we also define the $(i, j)$-*symbol graph* $G^{\text{sym}}_{ij}(L)$ as the $(i, j)$-row graph of its conjugate $L^{(13)}$. Any of these graphs are referred as a *two-line graph* of $L$.

Extending Latin squares terminology, a *line* in a partial Latin rectangle $L \in \mathrm{PLR}(r, s, n)$ is any set of all entries in $E(L)$ having the same value in a given component. Thus, a line is associated with either a row, a column or a symbol. In (4.1), an example of a line is $\{(1, 5, 6), (2, 4, 6)\}$ associated with the symbol 6.

Since two-line graphs are bipartite graphs with maximum degree at most 2, their connected components are either isolated vertices, paths or even-length cycles. We define the *length* of a path as its number of edges, including isolated vertices as paths of length 0. We classify the possible components in the following lemma.

**Lemma 3.** *Each connected component of a two-line graph has alternating edge colors (solid vs. dashed) and is isomorphic to a graph which is described by one of the following:*

$\mathfrak{w}_\ell$: *A path of even length $\ell \geq 0$ with white endpoints. If $\ell = 0$, then this is a white isolated vertex.*
$\mathfrak{b}_\ell$: *A path of even length $\ell \geq 0$ with black endpoints. If $\ell = 0$, then this is a black isolated vertex.*
$\mathfrak{s}_\ell$: *A path of odd length $\ell \geq 1$ with solid end-edges.*
$\mathfrak{d}_\ell$: *A path of odd length $\ell \geq 1$ with dashed end-edges.*
$\mathfrak{c}_\ell$: *A cycle of even length $\ell \geq 4$.*

As a consequence, unlike with arbitrary graphs [21] or general bipartite graphs [24–26,29,35,50], the task of enumerating cycles and paths in two-line graphs has low complexity.

For a two-line graph $G$, we define a frequency array

$$\begin{pmatrix} \mathfrak{w}_0 & \mathfrak{w}_2 & \mathfrak{w}_4 & \cdots \\ \mathfrak{b}_0 & \mathfrak{b}_2 & \mathfrak{b}_4 & \cdots \\ \mathfrak{s}_1 & \mathfrak{s}_3 & \mathfrak{s}_5 & \cdots \\ \mathfrak{d}_1 & \mathfrak{d}_3 & \mathfrak{d}_5 & \cdots \\ & \mathfrak{c}_4 & \mathfrak{c}_6 & \cdots \end{pmatrix}, \tag{4.2}$$

where each coordinate is the number of connected components of $G$ isomorphic to the graph described in Lemma 3. We write this as a sequence by reading the array column by column, giving

$$\sigma(G) := (\mathfrak{w}_0, \mathfrak{b}_0, \mathfrak{s}_1, \mathfrak{d}_1, \mathfrak{w}_2, \mathfrak{b}_2, \mathfrak{s}_3, \mathfrak{d}_3, \mathfrak{c}_4, \mathfrak{w}_4, \mathfrak{b}_4, \mathfrak{s}_5, \mathfrak{d}_5, \mathfrak{c}_6, \ldots),$$

which we call the *IC sequence* of $G$ (where IC is short for isomorphism class). For example, the IC sequence of the two-row graph of (4.1) is $(1, 0, 0, 1, 0, 1, 0, 1)$, after truncating the trailing zeros. The *length* of an IC sequence is the number of terms up to and including the last non-zero term.

The following lemma (which follows from Lemma 3) implies that the IC sequences of all the two-row graphs for a partial Latin rectangle in $\mathrm{PLR}(r, s, n)$ are determined by the rows $i$ and $j$ which satisfy $i < j$.

**Lemma 4.** *Let $L \in \mathrm{PLR}(r, s, n)$. For distinct $i, j \in [r]$, if*

$$\sigma(G_{ij}^{\mathrm{row}}(L)) = (\mathfrak{w}_0, \mathfrak{b}_0, \mathfrak{s}_1, \mathfrak{d}_1, \ldots)$$

*and*

$$\sigma(G_{ji}^{\mathrm{row}}(L)) = (\mathfrak{w}'_0, \mathfrak{b}'_0, \mathfrak{s}'_1, \mathfrak{d}'_1, \ldots),$$

*then $\mathfrak{b}'_k = \mathfrak{w}_k$, $\mathfrak{w}'_k = \mathfrak{b}_k$, $\mathfrak{s}'_k = \mathfrak{s}_k$, $\mathfrak{d}'_k = \mathfrak{d}_k$ and $\mathfrak{c}'_k = \mathfrak{c}_k$, for all $k$.*

The IC sequence of a two-line graph determines its isomorphism class, and thus determining whether two two-line graphs are isomorphic has low complexity. More specifically, the following proposition shows that we need only linearly many comparison operations.

**Proposition 3.** *The length of an IC sequence corresponding to a two-line graph on $2l$ vertices (where $l \geq 1$) is at most $5l - 1$.*

**Proof.** A two-row graph on $2l$ vertices has at most $2l$ edges. Thus, the corresponding IC sequence only has zeros after the term $\mathfrak{c}_{2l}$, which is the $(5l-1)$-th component of the IC sequence, as illustrated by (4.2). □

**Corollary 1.** *The problem of determining whether two IC sequences are equal requires at most $5l - 1$ pairwise comparisons, where $l$ is the number of entries in the largest line among the four lines that define the two corresponding two-line graphs.*

Two-line graphs enable us to introduce three new autotopism invariants that characterize pairs of rows, pairs of columns and pairs of symbols of a partial Latin rectangle. For $L \in PLR(r, s, n)$ and $i, j, i', j' \in [r]$ with $i \neq j$ and $i' \neq j'$, we write $(i, j) \sim_{\mathrm{row}} (i', j')$ to indicate that $G_{ij}^{\mathrm{row}}(L)$ and $G_{i'j'}^{\mathrm{row}}(L)$ are isomorphic, and define $\sim_{\mathrm{col}}$ and $\sim_{\mathrm{sym}}$ analogously. If $\overline{D}(t) := \{(p, p') \in [t] \times [t] : p \neq p'\}$, then the relations $\sim_{\mathrm{row}}$, $\sim_{\mathrm{col}}$ and $\sim_{\mathrm{sym}}$ constitute equivalence relations on the sets $\overline{D}(r)$, $\overline{D}(s)$ and $\overline{D}(n)$, respectively, for any given partial Latin rectangle, and thus define partitions on these sets. These partitions are $\mathrm{Atop}(L)$-invariant by the following theorem.

**Theorem 3.** *Let $L \in \mathrm{PLR}(r, s, n)$ and $(\alpha, \beta, \gamma) \in \mathrm{Atop}(L)$. Let $i, j \in \overline{D}(r)$, $i', j' \in \overline{D}(s)$ and $i'', j'' \in \overline{D}(n)$. Then $(i, j) \sim_{\mathrm{row}} (\alpha(i), \alpha(j))$, and $(i', j') \sim_{\mathrm{col}} (\beta(i'), \beta(j'))$, and $(i'', j'') \sim_{\mathrm{sym}} (\gamma(i''), \gamma(j''))$.*

**Proof.** We prove the statement for rows. The statements for columns and symbols follow symmetrically by considering $L^{(12)}$ and $L^{(13)}$. Consider the isotopisms $\theta = (\mathrm{Id}, \beta, \gamma)$ and $\theta' = (\alpha, \mathrm{Id}, \mathrm{Id})$, both in $S_r \times S_s \times S_n$. Since any permutation of columns or symbols of $L$ preserves the IC sequences of $G_{ij}^{\mathrm{row}}(L)$, the graphs $G_{ij}^{\mathrm{row}}(L^\theta)$ and $G_{ij}^{\mathrm{row}}(L)$ are isomorphic. Since $(L^\theta)^{\theta'} = L^{(\alpha,\beta,\gamma)} = L$, we have equality of graphs:

$$G_{ij}^{\mathrm{row}}(L^\theta) = G_{\alpha(i)\alpha(j)}^{\mathrm{row}}((L^\theta)^{\theta'}) = G_{\alpha(i)\alpha(j)}^{\mathrm{row}}(L),$$

and thus, $G_{ij}^{\mathrm{row}}(L) \cong G_{\alpha(i)\alpha(j)}^{\mathrm{row}}(L)$. $\square$

We define the *two-row representation of* $L \in \mathrm{PLR}(r, s, n)$ as the $r \times r$ matrix $\mathcal{R}_{\mathrm{row}}(L) = (r_{ij})$ which

(a) has an all-0 main diagonal,
(b) whose off-diagonal elements are positive integers such that $r_{ij} = r_{kl}$ if and only if $(i, j) \sim_{\mathrm{row}} (k, l)$, and
(c) is the lexicographically minimum such matrix (when read left to right, then top to bottom).

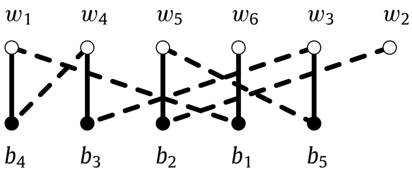
For the Latin square $L$ in (2.1) in Section 2, we compute

$$\mathcal{R}_{\mathrm{row}}(L) = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 1 & 4 \\ \hline 5 & 0 & 5 & 6 & 7 & 8 \\ \hline 9 & 1 & 0 & 10 & 1 & 11 \\ \hline 12 & 6 & 8 & 0 & 6 & 5 \\ \hline 5 & 7 & 5 & 6 & 0 & 13 \\ \hline 4 & 10 & 14 & 1 & 15 & 0 \\ \hline \end{array}. \tag{4.3}$$

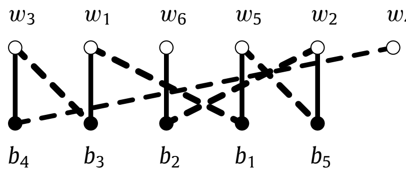
In this example, if we compare the pair of rows 2 and 1 in the Latin square $L$ with the pair of rows 5 and 1, we obtain the following two-row graphs

| | | | | | | |
|---|---|---|---|---|---|---|
| row 2 | 1 | 4 | 5 | 6 | 3 | 2 |
| row 1 | 4 | 3 | 2 | 1 | 5 | · |

| | | | | | | |
|---|---|---|---|---|---|---|
| row 5 | 3 | 1 | 6 | 5 | 2 | 4 |
| row 1 | 4 | 3 | 2 | 1 | 5 | · |




$G_{21}^{\mathrm{row}}(L)$

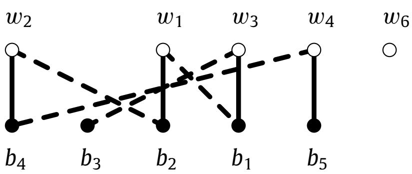



$G_{51}^{\mathrm{row}}(L)$

Thus $G_{21}^{\mathrm{row}}(L)$ is isomorphic to $G_{51}^{\mathrm{row}}(L)$. Thus, in the two-row representation $\mathcal{R}_{\mathrm{row}}(L) = (r_{ij})$ of $L$, we have $r_{21} = r_{51}$. In contrast, the two-row graph $G_{31}^{\mathrm{row}}(L)$ is determined from

| | | | | | | |
|---|---|---|---|---|---|---|
| row 3 | 2 | · | 1 | 3 | 4 | 6 |
| row 1 | 4 | 3 | 2 | 1 | 5 | · |

and is given by




which is not isomorphic to $G_{21}^{\mathrm{row}}(L)$, and thus $r_{21} \neq r_{31}$.

The *two-column* and *two-symbol representations* of $L$ are then analogously defined as the $s \times s$ and $n \times n$ arrays

$$\mathcal{R}_{\mathrm{col}}(L) := \mathcal{R}_{\mathrm{row}}(L^{(12)}) \qquad \text{and} \qquad \mathcal{R}_{\mathrm{sym}}(L) := \mathcal{R}_{\mathrm{row}}(L^{(13)}),$$

respectively. Any of these representations is referred as a *two-line representation* of $L$. For the Latin square $L$ in (2.1), we compute

$$\mathcal{R}_{\mathrm{col}}(L) = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 3 & 2 \\ \hline 4 & 0 & 5 & 6 & 6 & 7 \\ \hline 6 & 5 & 0 & 8 & 9 & 10 \\ \hline 3 & 2 & 11 & 0 & 12 & 1 \\ \hline 3 & 2 & 13 & 12 & 0 & 14 \\ \hline 6 & 15 & 16 & 4 & 17 & 0 \\ \hline \end{array}, \quad \mathcal{R}_{\mathrm{sym}}(L) = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 5 & 4 & 4 & 3 \\ \hline 1 & 5 & 0 & 4 & 4 & 6 \\ \hline 7 & 8 & 8 & 0 & 9 & 10 \\ \hline 11 & 8 & 8 & 12 & 0 & 13 \\ \hline 8 & 11 & 14 & 10 & 15 & 0 \\ \hline \end{array}. \tag{4.4}$$

The next observation follows readily from Lemma 4; it describes a property of two-row graphs $G_{ij}^{\text{row}}(L)$ when we swap the rows $i$ and $j$, and gives a correspondence between the elements in $\mathcal{R}_{\text{row}}(L)$ defined by taking its transpose.

**Proposition 4.** *For a partial Latin rectangle $L \in \text{PLR}(r, s, n)$ and $i, j, k, l \in [r]$ with $i \neq j$ and $k \neq l$, if $G_{ij}^{\text{row}}(L) \cong G_{kl}^{\text{row}}(L)$, then $G_{ji}^{\text{row}}(L) \cong G_{lk}^{\text{row}}(L)$, and likewise for the other two-line graphs.*

*Thus, in the two-row representation of $\mathcal{R}_{\text{row}}(L)$, if $r_{ij} = r_{kl}$, then $r_{ji} = r_{lk}$, and likewise for the other two-line representations.*

The following corollary to Theorem 3 defines row, column, and symbol invariants from a two-line representation.

**Corollary 2.** *Let $L \in \text{PLR}(r, s, n)$ and $(\alpha, \beta, \gamma) \in \text{Atop}(L)$.*

- *For any $i \in [r]$, rows $i$ and $\alpha(i)$ of $\mathcal{R}_{\text{row}}(L)$ are equal as multisets.*
- *For any $j \in [s]$, rows $j$ and $\beta(j)$ of $\mathcal{R}_{\text{col}}(L)$ are equal as multisets.*
- *For any $k \in [n]$, rows $k$ and $\gamma(k)$ of $\mathcal{R}_{\text{sym}}(L)$ are equal as multisets.*

For the Latin square $L$ in (2.1), since the rows of each of $\mathcal{R}_{\text{row}}(L)$ (as in (4.3)), and $\mathcal{R}_{\text{col}}(L)$ and $\mathcal{R}_{\text{sym}}(L)$ (as in (4.4)) are all distinct as multisets, Corollary 2 implies that $\text{Atop}(L)$ is trivial. As an example which admits a non-trivial autotopism, we investigate the following member of PLR(3, 3, 5; 7):

$$M := \begin{array}{|c|c|c|}\hline 1 & 2 & \cdot \\ \hline 2 & 1 & \cdot \\ \hline 3 & 4 & 5 \\ \hline \end{array}; \qquad \mathcal{R}_{\text{row}}(M) = \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 1 & 0 & 2 \\ \hline 3 & 3 & 0 \\ \hline \end{array} \qquad \begin{array}{l} \text{row 1 multiset: } \{0, 1, 2\}, \\ \text{row 2 multiset: } \{0, 1, 2\}, \\ \text{row 3 multiset: } \{0, 3, 3\}. \end{array},$$

The partial Latin rectangle $M$ admits the autotopism $((12), (12), (34)) \in S_3 \times S_3 \times S_5$, so Corollary 2 implies rows 1 and 2 of $\mathcal{R}_{\text{row}}(M)$ are equal as multisets, which we see in the above example.

We consider the complexity of computing $\mathcal{R}_{\text{row}}(L)$. We precede this with a technical lemma.

**Lemma 5.** *Let $(S_i)_{i=1}^u$ be a sequence of integer vectors $S_i = (S_i[j])_{j=1}^k$ satisfying $0 \leq S_i[j] \leq d_j - 1$ for all $j \in \{1, 2, \ldots, k\}$. In time $O(u \log(\prod_{j=1}^k d_j))$, it is possible to compute a sequence of integers $X = (x_i)_{i=1}^u$ such that*

*(1) the set (excluding repetitions) $\{x_1, x_2, \ldots, x_u\} = \{1, 2, \ldots, m\}$ for some $m \leq |X|$;*
*(2) the first occurrence of $i$ in $X$ occurs before the first occurrence of $i + 1$ in $X$; and*
*(3) $x_i = x_j$ if and only if $S_i = S_j$.*

**Proof.** Algorithm 1 describes the tree-based algorithm that we use to construct $X$.

In order to improve the search performance of Line 8 of Algorithm 1, for each node $v$ at depth $j$ we create a so-called *red–black tree* [23] (that is, a self-balancing binary search tree, which is widely used in Computer Science for searching in data structure) whose nodes are associated to the children of $v$. Performing the search for a particular child of $v$ (or finding that none exist with the new label $S_i[j]$) takes $O(\log d_j)$ time. Similarly, we update the red–black tree associated to vertex $v$ when we run Line 12 in Algorithm 1. This update also takes $O(\log d_j)$ time. Summing this over all $k$ depths results in $\sum_{j=1}^k O(\log d_j) = O(\log(\prod_{j=1}^k d_j))$. Thus, Lines 7 through 16 of Algorithm 1 require a total of $O(\log(\prod_{j=1}^k d_j))$ node label inspections (within the main tree and the red–black trees) and the creation of a total of $O(\log(\prod_{j=1}^k d_j))$ nodes (within the main tree and the red–black trees). Performing this over all $i \in [u]$ thus requires $O(u \log(\prod_{j=1}^k d_j))$ time, completing the proof. $\square$

**Theorem 4.** *The complexity of computing the three two-line representations of a partial Latin rectangle $L \in \text{PLR}(r, s, n)$ is $O(M^3)$, where $M = \max(r, s, n)$.*

**Proof.** We use $L$ stored internally in a matrix format. We pre-compute $L^{(23)}$, which is performed in time $O(rs)$.

We first analyze the complexity of computing the IC sequence of a pair of distinct rows $i, j \in [r]$. We start with an "uninspected" entry in row $i$ or $j$ as a length-0 path, say $(i, k, L[i, k])$. There may be a solid edge to the entry $(j, k, L[j, k])$, or there may be a dashed edge to the entry $(j, k', L[j, k'])$ where $k' = L^{(23)}[j, L[i, k]]$ (which we have pre-computed), or we may have both edges (or neither). In this way, we extend the endpoints of the path until it is no longer possible to extend it further. In the end, we identify the component $H$ containing the entry $(i, k, L[i, k])$: it is either a path or a cycle as listed in Lemma 3 and we mark all of $H$'s entries as "inspected". Consequently, we increase the count for $H$ in the IC sequence by 1. We then proceed to the next "uninspected" entry in row $i$ or $j$ until all entries in these rows are "inspected".

Performing this for rows $i$ and $j$ requires $O(s)$ time (such as when checking if $L[i, k]$ or $L[j, k]$ are defined, updating the subgraph counts, and updating the current path length; for brevity, we omit a complete accounting of memory usage). Doing so for each pair of distinct rows thus requires $O(r^2 s)$ time.

Having computed the $u := r^2 - r$ IC sequences, which we denote $S_i$ for $i \in \{1, 2, \ldots, u\}$, we place them into a vector $(S_i)_{i=1}^u$. We delete trailing zeros so that each $S_i$ has length $k := 5s - 1$, as per Proposition 3.

In Table 1, we tabulate the maximum frequency with which each possible component ($\mathfrak{w}_\ell$, $\mathfrak{b}_\ell$, $\mathfrak{s}_\ell$, $\mathfrak{d}_\ell$, and $\mathfrak{c}_\ell$ in Lemma 3) occurs in a two-row graph for an $s$-column partial Latin rectangle.

**Algorithm 1:** Generating $X$ from $\mathcal{S}$.

**Require:** Vectors $S_i \in [d_1] \times [d_2] \times \cdots \times [d_k]$, for all $i \in [u]$
**Ensure:** A sequence $X$ satisfying the conditions in the statement of Lemma 5

Initialize a rooted tree with a single (unlabeled) root node
Initialize $X$ as the length-$u$ all-0 vector
$c \leftarrow 1$
**for** $i$ from 1 to $u$ **do**
    Set $t$ as the root node
    $j \leftarrow 0$
    **while** $j < k$ **do**
        Search for a child node $d$ of $t$ with the label $S_i[j]$
        **if** $d$ found **then** ▷ consistent extension found in tree
            $t \leftarrow d$.
        **else** ▷ create consistent extension in tree
            Create a child node for $t$ labeled $S_i[j]$
            Set $t$ as this newly created child node
        **end if**
        $j \leftarrow j + 1$
    **end while**
    **if** leaf node $t$ is newly created **then** ▷ $S_i$ has not been seen before
        $x_i \leftarrow c$ ▷ add new value to ordered set of integers
        $w(t) \leftarrow c$ ▷ store leaf-node values
        $c \leftarrow c + 1$
    **else** ▷ $S_i$ has been seen before
        $x_i \leftarrow w(t)$ ▷ add old value to ordered set of integers
    **end if**
**end for**

**Table 1**
The theoretical maximum number of components of the various types in a two-row graph for an $s$-column partial Latin rectangle. Components which are not accounted for in the table cannot arise.

| | Maximum frequency | Valid for |
|---|---|---|
| $\mathfrak{w}_\ell$ | $\lfloor 2s/(\ell+2) \rfloor$ | even $\ell \geq 0$ |
| $\mathfrak{b}_\ell$ | $\lfloor 2s/(\ell+2) \rfloor$ | even $\ell \geq 0$ |
| $\mathfrak{s}_\ell$ | $\lfloor 2s/(\ell+1) \rfloor$ | odd $\ell \geq 1$ |
| $\mathfrak{d}_\ell$ | $\lfloor 2s/(\ell+1) \rfloor$ | odd $\ell \geq 1$ |
| $\mathfrak{c}_\ell$ | $\lfloor 2s/\ell \rfloor$ | even $\ell \geq 4$ |

Consequently, each sequence $S_i$ satisfies $0 \leq S_i[j] \leq \lfloor 5s/j \rfloor - 1$ for all $j \in \{1, 2, \ldots, k\}$, to which we apply Lemma 5 to obtain $X = (x_i)_{i=1}^{u}$. Using $X$, we fill cell $(i, j)$ of $\mathcal{R}_{\text{row}}(L)$ with the integer corresponding to the IC-sequence of rows $i$ and $j$ when $i \neq j$, and with 0 when $i = j$.

The number of operations required to compute $X$ by Lemma 5 is

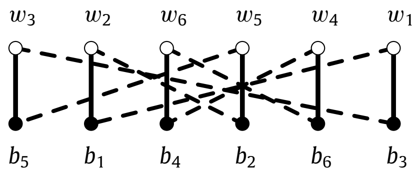
$$\begin{aligned} O\big(u \log(\textstyle\prod_{j=1}^{k} d_j)\big) &= O\big((r^2 - r)\log(\textstyle\prod_{j=1}^{5s} 5s/j)\big) \\ &= O((r^2 - r)\log((5s)^{5s}/(5s)!)) \\ &= O((r^2 - r)(5s - O(\log(5s)))) \qquad \text{using Stirling's approximation} \\ &= O(r^2 s). \end{aligned}$$

Thus, computing $\mathcal{R}_{\text{row}}(L)$ requires $O(r^2 s) \leq O(M^3)$ time. Similarly, computing $\mathcal{R}_{\text{col}}(L)$ and $\mathcal{R}_{\text{col}}(L)$ also has complexity at most $O(M^3)$. □

**Remark 1.** In the case of a (full) Latin rectangle the two-line graphs correspond to the cycle representation of permutations in the following way. Let $i$ and $j$ be two rows of a given Latin rectangle $L$ (or two columns or two symbols) and let $\sigma_i$ and $\sigma_j$ be the corresponding row permutations. Then, the connected components of $G_{ij}^{\text{row}}(L)$ are all cycles and they correspond to the cycles in the cycle representation of $\sigma_j \sigma_i^{-1}$ (applied from right to left). For example, let

$$L = \begin{array}{|c|c|c|c|c|c|} \hline 3 & 2 & 6 & 5 & 4 & 1 \\ \hline 5 & 1 & 4 & 2 & 6 & 3 \\ \hline \end{array}$$

where $\sigma_2\sigma_1^{-1} = (1352)(46)$. These cycles can be obtained from the cycles in two-row graph




by leaving just the indexes of the white vertices.

The method introduced in [34] for computing the autotopism group of a Latin square is based on the cycle structure of permutations of the form $\sigma_j\sigma_i^{-1}$, where $\sigma_i$ and $\sigma_j$ are two row permutations. Thus, the refinement method described in the next section generalizes [34] to partial Latin rectangles.

## 5. The two-line graph refinement

We use two-line representations to define a partition refinement method. Let $A \subseteq [r]$ and $L \in \mathrm{PLR}(r, s, n)$. Let $\mathcal{R}_{\mathrm{row}}(L) = (\rho[i, j])$ and define the multiset

$$\rho[i, A] = \{\rho[i, j]\}_{j\in A}.$$

(As a memory aid, $\rho$ is pronounced “row”.) For example, for the two-row representation $\mathcal{R}_{\mathrm{row}}(L)$ in (4.3), we have $\rho[1, \{3, 6\}] = \{2, 4\}$.

**Proposition 5.** *Let $L \in \mathrm{PLR}(r, s, n)$, let $\mathcal{P}$ be a row partition that is fixed by $\mathrm{Atop}(L)$, and let $(\alpha, \beta, \gamma) \in \mathrm{Atop}(L)$. Then $\rho[i, P] = \rho[\alpha(i), P]$ for every $i \in [r]$ and $P \in \mathcal{P}$.*

**Proof.** By definition, the multiset

$$\begin{aligned}\rho[i, P] &= \{\rho[i, j]\}_{j\in P} \\ &= \{\rho[\alpha(i), \alpha(j)]\}_{j\in P} && \text{by Theorem 3} \\ &= \{\rho[\alpha(i), j]\}_{j\in P} && \text{since } \mathcal{P} \text{ is fixed by } \mathrm{Atop}(L)\text{, so } P = \alpha^{-1}(P) \\ &= \rho[\alpha(i), P]. \quad \square\end{aligned}$$

The significance of Proposition 5 is applying its contrapositive, which is the following corollary. It gives a condition on when there exists an autotopism which maps row $r_1$ to row $r_2$.

**Corollary 3.** *Let $L \in \mathrm{PLR}(r, s, n)$ and let $i, j \in [r]$. If for some $\mathrm{Atop}(L)$-fixed row partition $\mathcal{P}$ and some part $P \in \mathcal{P}$ the multisets $\rho[i, P]$ and $\rho[j, P]$ are distinct, then no autotopism of $L$ maps row $i$ to $j$. In other words, $i$ and $j$ are in different parts of the row partition in the orbits system $\mathscr{P}_O(L)$.*

**Definition 2.** Let $L \in \mathrm{PLR}(r, s, n)$. The *two-line graph (TLG) refinement* is the map

$$G : \mathrm{SPart}(L) \to \mathrm{SPart}(L)$$

where $G(\mathscr{P}) := (G_{\mathrm{row}}(\mathcal{P}_{\mathrm{row}}), G_{\mathrm{col}}(\mathcal{P}_{\mathrm{col}}), G_{\mathrm{sym}}(\mathcal{P}_{\mathrm{sym}}))$, for $\mathscr{P} = (\mathcal{P}_{\mathrm{row}}, \mathcal{P}_{\mathrm{col}}, \mathcal{P}_{\mathrm{sym}}) \in \mathrm{SPart}(L)$, is defined as follows. Two distinct rows $i$ and $j$ belong to the same part of $G_{\mathrm{row}}(\mathcal{P}_{\mathrm{row}})$ if and only if

(1) $i \sim_{\mathcal{P}_{\mathrm{row}}} j$, and
(2) $\rho[i, P] = \rho[j, P]$, for every $P \in \mathcal{P}_{\mathrm{row}}$.

The maps $G_{\mathrm{col}}$ and $G_{\mathrm{sym}}$ are defined analogously using the two-column and two-symbol representations $\mathcal{R}_{\mathrm{col}}(L)$ and $\mathcal{R}_{\mathrm{sym}}(L)$, respectively.

**Theorem 5.** *Let $L \in \mathrm{PLR}(r, s, n)$. The two-line graph refinement $G$ on $\mathrm{SPart}(L)$ satisfies the following properties:*

*(i) $G$ is a refinement.*
*(ii) $G$ is a map from $\mathrm{SPart}_{\mathrm{Atop}}(L)$ to itself.*
*(iii) $G$ is order-preserving.*
*(iv) $G$ is a meet-homomorphism.*

**Proof.** The first statement holds from the definition of $G$. The second follows from Proposition 5. For (iii) and (iv), let $\mathscr{P} = (\mathcal{P}_{\mathrm{row}}, \mathcal{P}_{\mathrm{col}}, \mathcal{P}_{\mathrm{sym}})$ and $\mathscr{P}' = (\mathcal{P}'_{\mathrm{row}}, \mathcal{P}'_{\mathrm{col}}, \mathcal{P}'_{\mathrm{sym}})$ be two systems of partitions in $\mathrm{SPart}(L)$. We prove each statement separately by focusing on the corresponding row partitions. (The proofs for the column and the symbol partitions follow similarly.)

(iii) Suppose $\mathscr{P}' \leq \mathscr{P}$. We need to show that $G(\mathscr{P}') \leq G(\mathscr{P})$. We prove that $G_{\text{row}}(\mathcal{P}'_{\text{row}}) \leq G_{\text{row}}(\mathcal{P}_{\text{row}})$, for which it is sufficient to show that $i \not\sim_{G_{\text{row}}(\mathcal{P}'_{\text{row}})} j$ whenever $i \not\sim_{G_{\text{row}}(\mathcal{P}_{\text{row}})} j$ for distinct $i, j \in [r]$. If $i \not\sim_{\mathcal{P}_{\text{row}}} j$, then $i \not\sim_{\mathcal{P}'_{\text{row}}} j$ (since $\mathcal{P}'_{\text{row}} \leq \mathcal{P}_{\text{row}}$), and thus $i \not\sim_{G(\mathcal{P}'_{\text{row}})} j$ by (i). Now suppose $i \sim_{\mathcal{P}_{\text{row}}} j$. By the definition of $G_{\text{row}}(\mathcal{P}_{\text{row}})$, the assumption that $i \not\sim_{G_{\text{row}}(\mathcal{P}_{\text{row}})} j$ implies that there exists a part $P \in \mathcal{P}_{\text{row}}$ such that $\rho[i, P] \neq \rho[j, P]$. Since $\mathcal{P}'_{\text{row}} \leq \mathcal{P}_{\text{row}}$, there exists $P' \subseteq P$, such that $P' \in \mathcal{P}'_{\text{row}}$ and $\rho[i, P'] \neq \rho[j, P']$. So, $i \not\sim_{G_{\text{row}}(\mathcal{P}'_{\text{row}})} j$.

(iv) The third statement implies readily that

$$G_{\text{row}}(\mathcal{P}_{\text{row}} \wedge \mathcal{P}'_{\text{row}}) \leq G_{\text{row}}(\mathcal{P}_{\text{row}}) \wedge G_{\text{row}}(\mathcal{P}'_{\text{row}}).$$

Now, if $i \sim_{G_{\text{row}}(\mathcal{P}_{\text{row}}) \wedge G_{\text{row}}(\mathcal{P}'_{\text{row}})} j$, for some $i, j \in [r]$, then Lemma 1(i) implies that $i \sim_{G_{\text{row}}(\mathcal{P}_{\text{row}})} j$ and $i \sim_{G_{\text{row}}(\mathcal{P}'_{\text{row}})} j$. Thus, from Definition 2, we have that:

(a) $i \sim_{\mathcal{P}_{\text{row}}} j$ and $i \sim_{\mathcal{P}'_{\text{row}}} j$. Hence, again from Lemma 1(i), we have that $i \sim_{\mathcal{P}_{\text{row}} \wedge \mathcal{P}'_{\text{row}}} j$.

(b) $\rho[i, P] = \rho[j, P]$, for all $P \in \{\mathcal{P}_{\text{row}}, \mathcal{P}'_{\text{row}}\}$, and hence, for all $P \in \mathcal{P}_{\text{row}} \wedge \mathcal{P}'_{\text{row}}$.

As a consequence, we have that $i \sim_{G_{\text{row}}(\mathcal{P}_{\text{row}} \wedge \mathcal{P}'_{\text{row}})} j$. $\square$

In order to illustrate that $G$ is not a join-homomorphism (and hence, not a lattice homomorphism), let us consider the partial Latin square of order four

$$L = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdot & \cdot \\ \hline \cdot & 1 & 2 & \cdot \\ \hline \cdot & \cdot & 1 & 2 \\ \hline 2 & \cdot & \cdot & 1 \\ \hline \end{array}$$

together with the two row partitions

$$\mathcal{P}_{\text{row}} = \{\{1, 2\}, \{3, 4\}\} \qquad \text{and} \qquad \mathcal{P}'_{\text{row}} = \{\{1\}, \{2, 3\}, \{4\}\}.$$

Then,

$$\mathcal{R}_{\text{row}}(L) = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 1 \\ \hline 1 & 0 & 1 & 2 \\ \hline 2 & 1 & 0 & 1 \\ \hline 1 & 2 & 1 & 0 \\ \hline \end{array}$$

and

$$\mathcal{P}_{\text{row}} \vee \mathcal{P}'_{\text{row}} = \{\{1, 2, 3, 4\}\}.$$

Notice that

$$G_{\text{row}}(\mathcal{P}_{\text{row}}) = \mathcal{P}_{\text{row}} \qquad \text{and} \qquad G_{\text{row}}(\mathcal{P}'_{\text{row}}) = \{\{1\}, \{2\}, \{3\}, \{4\}\}.$$

As a consequence,

$$G_{\text{row}}(\mathcal{P}_{\text{row}}) \vee G_{\text{row}}(\mathcal{P}'_{\text{row}}) = \mathcal{P}_{\text{row}} < \{\{1, 2, 3, 4\}\} = G_{\text{row}}(\mathcal{P}_{\text{row}} \vee \mathcal{P}'_{\text{row}}).$$

The following result deals with the composition of both natural and two-line graph refinements.

**Corollary 4.** *Let $L \in \mathrm{PLR}(r, s, n)$ and $\mathscr{P}$ be a system of partitions that is fixed by* $\mathrm{Atop}(L)$*. Then $N(G(\mathscr{P})) \leq \mathscr{P}_{\mathrm{SEI}}(L)$.*

**Proof.** By symmetry, it is sufficient to show that for distinct $i, j \in [r]$ such that $i \sim_{G_{\text{row}}(\mathcal{P}_{\text{row}})} j$, the rows $i$ and $j$ have the same number of entries; the claim then follows by the definition of SEIs and Theorem 1. If these rows have a distinct number of entries, then the graphs that define row $i$ of $\mathcal{R}_{\text{row}}(L)$ have a different number of white vertices than the graphs that define row $j$ of $\mathcal{R}_{\text{row}}(L)$, and are thus non-isomorphic. This implies that rows $i$ and $j$ of $\mathcal{R}_{\text{row}}(L)$ contain different multisets of symbols. $\square$

Let $\mathscr{P} \in \mathrm{SPart}(L)$. We apply $G$ repeatedly, that is

$$\cdots \leq G^k(\mathscr{P}) \leq \cdots \leq G^2(\mathscr{P}) \leq G(\mathscr{P}) \leq \mathscr{P}, \tag{5.1}$$

until no more refinement is achieved. We denote the finest system of partitions $G^{\infty}(\mathscr{P})$ and call it the *exhaustive two-line graph refinement*, or exhaustive TLG refinement, of $\mathscr{P}$. An inductive argument based on Theorem 5 shows that for any two systems of partitions $\mathscr{P}$ and $\mathscr{P}'$, if $\mathscr{P}' \leq \mathscr{P}$, then $G^{\infty}(\mathscr{P}') \leq G^{\infty}(\mathscr{P})$.

Notice that the family $\{G^{\infty}(\mathscr{P}) \colon \mathscr{P} \in \mathrm{SPart}(L)\}$ constitutes the set of fixed points of the order-preserving map $G$ in the complete lattice $\mathrm{SPart}(L)$. Then, Tarski's fixed point theorem [49] implies that such a family also forms a complete lattice. The same holds for the family $\{G^{\infty}(\mathscr{P}) \colon \mathscr{P} \in \mathrm{SPart}_{\mathrm{Atop}}(L)\}$. The maxima and minima of both complete lattices coincide with those of $\mathrm{SPart}(L)$ and $\mathrm{SPart}_{\mathrm{Atop}}(L)$, which were indicated after Lemma 1.

We now approximate the complexity of computing $G^\infty(\mathscr{P})$. The procedure consists of two parts: (i) computing the matrices $\mathcal{R}_{\mathrm{row}}(L)$, $\mathcal{R}_{\mathrm{col}}(L)$ and $\mathcal{R}_{\mathrm{sym}}(L)$, for which the complexity is resolved by Theorem 4, and (ii) performing the iterations in (5.1). The time complexity of each iteration is dominated by multiset comparisons, which is where we focus our attention.

**Theorem 6.** *Let $\mathscr{P}$ be a system of partitions on $L \in \mathrm{PLR}(r, s, n)$ and let $M = \max(r, s, n)$. The complexity of computing $G(\mathscr{P})$ is at most $O(M^3)$ and the complexity of computing $G^\infty(\mathscr{P})$ is at most $O(M^3 \log M)$.*

**Proof.** Note that the iterations in (5.1) can be performed independently for rows, columns and symbols, so we focus on calculating the row component of $G^\infty(\mathscr{P})$. Theorem 4 implies $\mathcal{R}_{\mathrm{row}}(\mathcal{P})$, $\mathcal{R}_{\mathrm{col}}(\mathcal{P})$ and $\mathcal{R}_{\mathrm{sym}}(\mathcal{P})$ is computable in time $O(M^3)$, so we assume these are pre-computed.

Let $\mathcal{P} = \{P_1, \dots, P_t\}$ be a row partition of $L \in \mathrm{PLR}(r, s, n)$. We analyze the complexity of computing $G_{\mathrm{row}}(\mathcal{P})$, that is, the complexity of subdividing each $P_i$ according to the definition of $G_{\mathrm{row}}(\mathcal{P})$. For every pair $i, j \in [t]$ (not necessarily distinct) let $\mathcal{R}^{i,j}_{\mathrm{row}}(L)$ be the submatrix of $\mathcal{R}_{\mathrm{row}}(L)$ formed by the intersection of the rows in $P_i$ and the columns in $P_j$.

As an example, we might have

$$L = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \cdot & \cdot & \cdot \\ \hline \cdot & 1 & 2 & \cdot & \cdot \\ \hline 2 & \cdot & 1 & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & 3 & 4 \\ \hline \cdot & \cdot & \cdot & \cdot & 5 \\ \hline \end{array}, \qquad \mathcal{R}_{\mathrm{row}}(L) = \begin{array}{|c|c|c||c|c|}\hline 0 & 1 & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 2 & 3 \\ \hline\hline 2 & 2 & 2 & 0 & 4 \\ \hline 5 & 5 & 5 & 6 & 0 \\ \hline \end{array},$$

and the row partition $\mathcal{P} = \{\{1, 2, 3\}, \{4, 5\}\}$.

To subdivide $P_i$ we first sort the indices of $P_i$ according to the multisets of symbols in rows of $\mathcal{R}^{i,1}_{\mathrm{row}}(L)$. We then subdivide $P_i$ according to whether these multisets are equal or not. Afterwards, we subdivide each new sub-part of $P_i$ by the multisets of entries in $\mathcal{R}^{i,2}_{\mathrm{row}}(L)$, and so on. The end result is a partition $P_i^*$ of $P_i$. Having computed $P_i^*$ for all $i \in [t]$, we have $G_{\mathrm{row}}(\mathcal{P}) = \cup_{i \in [t]} P_i^*$.

In the above example, the rows of the submatrix $\mathcal{R}^{2,1}_{\mathrm{row}}(L)$ give the multisets $\{2, 2, 2\}$ and $\{5, 5, 5\}$; they are distinct, which implies that rows 4 and 5 do not map to one another in any autotopism of $L$, and we subdivide the part $P_2 = \{4, 5\}$ into two parts $\{4\}$ and $\{5\}$, giving $P_2^* = \{\{4\}, \{5\}\}$. We also compute $P_1^* = \{\{1, 2, 3\}\}$, and thus $G_{\mathrm{row}}(\mathcal{P}) = \{\{1, 2, 3\}, \{4\}, \{5\}\}$.

Thus, the complexity of subdividing each $P_i$ is bounded by the complexity of sorting the rows (according to their multisets of entries) in all the submatrices $\mathcal{R}^{i,j}_{\mathrm{row}}(L)$, $j \in [t]$, and hence, the complexity of subdividing $\mathcal{P}$ is bounded by the complexity of sorting the rows in all the submatrices $\mathcal{R}^{i,j}_{\mathrm{row}}(L)$ where $i, j \in [t]$.

We sort the symbols in each row of each submatrix $\mathcal{R}^{i,j}_{\mathrm{row}}(L)$: there are $|P_i|$ rows to sort, with each row taking $O(|P_j| \log |P_j|)$ time. We also sort the rows in each $\mathcal{R}^{i,j}_{\mathrm{row}}(L)$, which takes $O(|P_i| \log |P_i|)$ row comparisons and where each row comparison uses at most $|P_j|$ symbol comparisons. As $\log |P_i| \le \log r$ and $\log |P_j| \le \log r$, summing over all submatrices yields $O(r^2 \log r)$.

As the process involves no more than $r$ refinement iterations, the complexity of computing the row component of $G^\infty(\mathscr{P})$ is at most $O(r^3 \log r) \le O(M^3 \log M)$. Similarly, performing the analogous computation for columns and symbols has complexity $O(M^3 \log M)$. $\square$

## 6. Examples

In this section, we illustrate the use of the techniques described in this paper. For the partial Latin rectangle

$$L = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1 & \cdot & 2 & \cdot & \cdot & \cdot & 3 & \cdot & \cdot \\ \hline 2 & \cdot & \cdot & 4 & 1 & 5 & 6 & \cdot & 7 \\ \hline \cdot & 1 & 5 & 3 & \cdot & 4 & \cdot & \cdot & \cdot \\ \hline \cdot & 2 & \cdot & 5 & \cdot & 3 & \cdot & 4 & \cdot \\ \hline 4 & 3 & \cdot & \cdot & 5 & \cdot & 1 & \cdot & 2 \\ \hline \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & 1 & 3 \\ \hline \end{array},$$

in $\mathrm{PLR}(6, 9, 7)$ we obtain $\mathscr{P}_{\mathrm{SEI}}(L) = (\mathcal{P}_{\mathrm{row}}, \mathcal{P}_{\mathrm{col}}, \mathcal{P}_{\mathrm{sym}})$, where

$$\mathcal{P}_{\mathrm{row}} = \{\{1, 6\}, \{2\}, \{3, 4\}, \{5\}\},$$
$$\mathcal{P}_{\mathrm{col}} = \{\{1, 5\}, \{2\}, \{3, 8\}, \{4, 6\}, \{7, 9\}\}, \text{ and}$$
$$\mathcal{P}_{\mathrm{sym}} = \{\{1, 2\}, \{3\}, \{4, 5\}, \{6, 7\}\}.$$

An exhaustive search yields $\mathrm{Atop} = \{\mathrm{Id}, \theta\}$ where

$$\theta = ((16)(34), (15)(38)(46)(79), (12)(45)(67)) \in S_6 \times S_9 \times S_7. \tag{6.1}$$

So, the system $\mathscr{P}_{\mathrm{SEI}}(L)$ is the orbits system. Next, we ordinarily search among the isomorphisms that fix $\mathscr{P}_{\mathrm{SEI}}(L)$ to deduce $\mathrm{Atop}(L)$, since not all isotopisms that preserve the orbit-system of partitions are autotopisms. In this case the two-line representations can save us the search. We compute

$$\mathcal{R}_{\mathrm{row}}(L)=(r_{ij})=\begin{array}{|c|c|c|c|c|c|}\hline 0&1&2&3&4&5\\\hline 1&0&4&4&6&1\\\hline 2&7&0&8&9&3\\\hline 3&7&8&0&9&2\\\hline 7&10&9&9&0&7\\\hline 5&1&3&2&4&0\\\hline\end{array}\,,\qquad \mathcal{R}_{\mathrm{sym}}(L)=\begin{array}{|c|c|c|c|c|c|c|}\hline 0&1&2&3&4&5&6\\\hline 1&0&2&4&3&6&5\\\hline 2&2&0&7&7&8&8\\\hline 9&4&10&0&2&6&6\\\hline 4&9&10&2&0&6&6\\\hline 11&6&8&6&6&0&6\\\hline 6&11&8&6&6&6&0\\\hline\end{array}\,,$$

and

$$\mathcal{R}_{\mathrm{col}}(L)=(c_{ij})=\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 0&1&2&2&3&4&5&6&7\\\hline 1&0&4&8&1&8&9&4&9\\\hline 10&4&0&10&6&10&11&12&13\\\hline 10&14&2&0&4&15&4&2&4\\\hline 3&1&6&4&0&2&7&2&5\\\hline 4&14&2&15&10&0&4&2&4\\\hline 5&9&11&4&7&4&0&13&16\\\hline 6&4&12&10&10&10&13&0&11\\\hline 7&9&13&4&5&4&16&11&0\\\hline\end{array}\,.$$

From the orbit-system of partitions, we cannot immediately deduce which of the three non-trivial permutations $(16)$, $(34)$ or $(16)(34)$ (if any) could be the first component of an autotopism of $L$. However, since $r_{13}\neq r_{63}$, we exclude the permutation $(16)$ as the row permutation in any element of $\mathrm{Atop}(L)$, and since $r_{13}\neq r_{14}$, we likewise exclude $(34)$. We also observe that $c_{1k}\neq c_{5k}$ for $k\in\{3,4,7\}$. Thus, if a column permutation $\beta$ in an autotopism of $L$ contains the transposition $(15)$, it cannot fix any of 3, 4 and 7, and hence, from the system $\mathscr{P}_{\mathrm{SEI}}(L)$ above, $\beta$ must contain the three transpositions $(38)$, $(46)$ and $(79)$. Further, since $c_{31}\neq c_{81}$, $c_{41}\neq c_{61}$ and $c_{71}\neq c_{91}$, we have that, if $\beta$ does not contain the transposition $(15)$, then it must be $\beta=\mathrm{Id}$. Thus, the only possible non-trivial $\beta$ in an autotopism $(\alpha,\beta,\gamma)\in\mathrm{Atop}(L)$ is $(15)(38)(46)(79)$. These restrictions on $\alpha$ and $\beta$ limit a non-trivial $(\alpha,\beta,\gamma)\in\mathrm{Atop}(L)$ to having the form $((16)(34),(15)(38)(46)(79),\gamma)$. Since $\gamma$ is determined by $\alpha$, $\beta$ and $L$ (as we exclude the existence of unused symbols), we deduce that $\theta$ in (6.1) is the only non-trivial autotopism of $L$ without searching.

We next illustrate that in some extreme cases the TLG refinement is useless. The following Latin square of order 11 is an example of an *atomic* Latin square from [36], that is, all its two-line graphs are single cycles, and are thus isomorphic.

$$L=\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 3&5&1&10&2&11&8&7&9&4&6\\\hline 5&4&6&2&1&3&11&9&8&10&7\\\hline 1&6&5&7&3&2&4&11&10&9&8\\\hline 10&2&7&6&8&4&3&5&11&1&9\\\hline 2&1&3&8&7&9&5&4&6&11&10\\\hline 11&3&2&4&9&8&10&6&5&7&1\\\hline 8&11&4&3&5&10&9&1&7&6&2\\\hline 7&9&11&5&4&6&1&10&2&8&3\\\hline 9&8&10&11&6&5&7&2&1&4&4\\\hline 4&10&9&1&11&7&6&8&3&2&5\\\hline 6&7&8&9&10&1&2&3&4&5&11\\\hline\end{array}\tag{6.2}$$

In this example, no refinement beyond the system of 1-part partitions $\mathscr{P}_1(L)$ is possible. However, $\mathscr{P}_1(L)$ is not the orbits system, as [36] states $|\mathrm{Atop}(L)|=10$. However, atomic Latin squares are extremely rare.

For the partial Latin rectangle

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \cdot&4&7&\cdot&\cdot&\cdot&\cdot&2&8\\\hline \cdot&\cdot&8&\cdot&\cdot&\cdot&2&4&3\\\hline 9&\cdot&1&6&5&\cdot&\cdot&\cdot&\cdot\\\hline 5&\cdot&6&a&\cdot&9&\cdot&\cdot&\cdot\\\hline \cdot&7&\cdot&\cdot&\cdot&\cdot&3&8&\cdot\\\hline \cdot&2&3&\cdot&\cdot&\cdot&4&\cdot&7\\\hline 6&\cdot&\cdot&\cdot&1&a&\cdot&\cdot&\cdot\\\hline \cdot&\cdot&a&1&9&5&\cdot&\cdot&\cdot\\\hline\end{array}$$

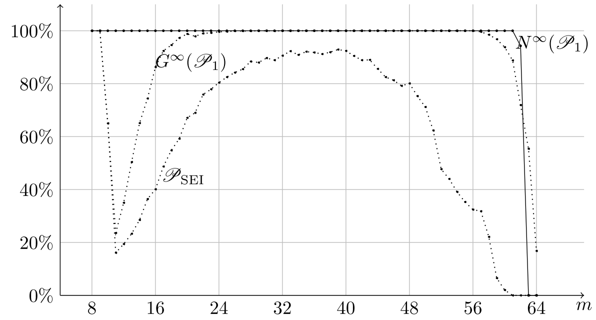



**Fig. 1.** Percentage of detection of orbit-system of partitions by the partition refinements $\mathscr{P}_{\mathrm{SEI}}$, $N^\infty(\mathscr{P}_1)$ and $G^\infty(\mathscr{P}_1)$ applied to 1000 random rectangles in PLR$(8, 8, 8; m)$, for each value of $m$. For all of them, the partition refinement $N^\infty(G^\infty(\mathscr{P}_1))$ gives the orbit-system of partitions 100% of the time.

where $a = 10$, the TLG refinement, combined with the exhaustive natural refinement, produces the orbits partitions of the rows and columns but not of the symbols. Its autotopism group has cardinality 18 and is generated by the three autotopisms

$((182364)(57), (126857)(49), (1367a8)(29)(45)),$
$((136824)(57), (125768)(49), (13a867)(29)(45))$, and
$((186423)(57), (185267)(49), (17a368)(29)(45)).$

In this case, both $N^\infty(G^\infty(\mathscr{P}_1))$ and $G^\infty(N^\infty(\mathscr{P}_1))$ yield the system defined by:

$$\mathcal{P}_{\mathrm{row}} = \{\{1, 2, 3, 4, 6, 8\}, \{5, 7\}\},$$
$$\mathcal{P}_{\mathrm{col}} = \{\{1, 2, 5, 6, 7, 8\}, \{3\}, \{4, 9\}\}, \text{ and}$$
$$\mathcal{P}_{\mathrm{sym}} = \{\{1, 3, 6, 7, 8, a\}, \{2, 4, 5, 9\}\}.$$

Thus, this is an example where the methods in this paper give row and column partitions that are equal to the orbits partitions, but the symbol partition is not.

## 7. Experimental results

We test the partition refinement methods described throughout the paper on randomly generated partial Latin rectangles. Randomization is performed in two ways:

(A) Starting with an empty $r \times s$ array $L$ and randomly adding triples $(i, j, k) \in [r] \times [s] \times [n]$ to Ent$(L)$, omitting illegal triples, until $m$ entries is reached.
(B) Starting with a randomly generated Latin rectangle (using the method of [28] to generate random Latin squares, then deleting rows and columns to obtain a PLR$(r, s, n; rs)$) and randomly deleting $rs - m$ entries.

In both methods, we added the additional condition that $L$ contains at least one entry in each row and in each column, and that each symbol appears at least once. Note that Latin rectangles generated with method B are not uniformly random, as they can always be completed to a Latin rectangle, a condition that does not hold for partial Latin rectangles in general. An implementation of both methods is available at http://plr.telhai.ac.il.

For each randomly generated partial Latin rectangle we compute four partition refinements: $N^2(\mathscr{P}_1)$ (equal to $\mathscr{P}_{\mathrm{SEI}}$), $N^\infty(\mathscr{P}_1)$ (equal to $N^\infty(\mathscr{P}_{\mathrm{SEI}})$), $G^\infty(\mathscr{P}_1)$ and $N^\infty(G^\infty(\mathscr{P}_1))$ (we use $\mathscr{P}_1$ (instead of $\mathscr{P}_1(L)$) to denote the system of 1-part partitions for the input PLR $L$). We measure the performance of each refinement as the proportion of Latin rectangles for which the resulting system equals the orbits system (the theoretical best possible outcome). In the first experiment, we generate 1000 squares in PLR$(8, 8, 8; m)$ for each $m \in \{8, \dots, 64\}$. After computing their respective partition refinements, we observe that the combined use of the exhaustive TLG refinement and the exhaustive natural refinement (that is, $N^\infty(G^\infty(\mathscr{P}_1))$) proves particularly useful – consistently giving the orbit-system of partitions 100% of the time – which is pertinent when the partial Latin rectangles are full or almost full. For the rest of partition refinements (that is, $\mathscr{P}_{\mathrm{SEI}}$, $N^\infty(\mathscr{P}_1)$ and $G^\infty(\mathscr{P}_1)$), the results are recorded in Fig. 1.

We also carry out the above-mentioned experiment for 1000 random squares in PLR$(8, 9, 10; m)$, for each $m \in \{10, \dots, 72\}$. In this case, both partition refinements $N^\infty(\mathscr{P}_1)$ and $N^\infty(G^\infty(\mathscr{P}_1))$ give the orbit-system of partitions 100% of the time. Fig. 2 shows the results for the partition refinements $\mathscr{P}_{\mathrm{SEI}}$ and $G^\infty(\mathscr{P}_1)$.

Regarding the difference in the performance of $N^\infty(\mathscr{P}_1)$ between the two experiments: for Latin squares, $\mathscr{P}_{\mathrm{SEI}} = \mathscr{P}_1$ and its natural refinements are also $\mathscr{P}_1$. Thus, $N^\infty(\mathscr{P}_1)$ is ineffective in those cases.

Since the vast majority of the randomly generated partial Latin rectangles have a trivial autotopism group, which may affect the results of the analysis mentioned above, we also repeat the experiments on random partial Latin rectangles of

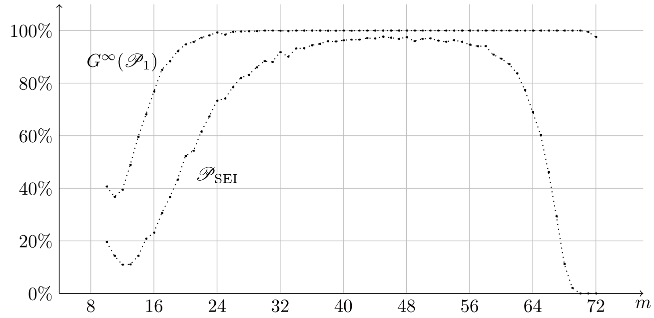



**Fig. 2.** Percentage of detection of orbit-system of partitions by the partition refinements $\mathscr{P}_{\mathrm{SEI}}$ and $G^\infty(\mathscr{P}_1)$ applied to 1000 random rectangles in PLR(8, 9, 10; $m$), for each value of $m$. For all of them, both partition refinements $N^\infty(\mathscr{P}_1)$ and $N^\infty(G^\infty(\mathscr{P}_1))$ give the orbit-system of partitions 100% of the time.

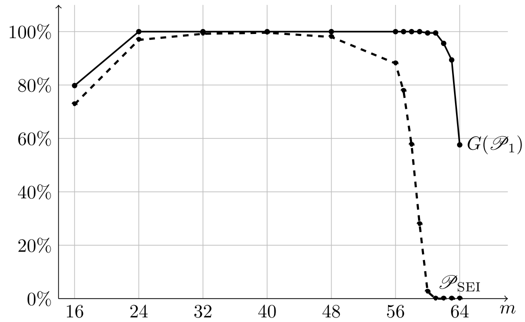



**Fig. 3.** Proportion of detection of the triviality of Atop($L$) for rectangles in PLR(8, 8, 8; $m$), by two refinement methods applied to 1000 random rectangles having trivial autotopism group, for each value of $m \in \{16, 24, 32, 40, 48, 56, 57, \dots, 64\}$.

**Table 2**
Percentage of cases in which the orbits system was obtained for random partial Latin squares with a non-trivial autotopism group.

| r | s | n | m | ♯Random PLRs | Optimal system of partitions (%) | |
|---|---|---|---|---|---|---|
| | | | | | $N^\infty(\mathscr{P}_1)$ | $G^\infty(\mathscr{P}_1)$ |
| 8 | 9 | 10 | 30 | 156 | 98.08 | 100.00 |
| 9 | 12 | 16 | 70 | 3265 | 82.45 | 97.55 |
| 9 | 14 | 14 | 70 | 4445 | 60.99 | 89.70 |
| 10 | 10 | 10 | 50 | 1206 | 79.02 | 95.19 |
| 12 | 18 | 18 | 90 | 8103 | 74.58 | 96.61 |
| 20 | 20 | 20 | 100 | 10473 | 93.79 | 99.45 |
| 20 | 20 | 20 | 120 | 5263 | 88.65 | 99.63 |
| 20 | 20 | 20 | 140 | 2094 | 88.20 | 99.47 |
| 20 | 20 | 20 | 160 | 692 | 83.67 | 99.42 |
| 20 | 20 | 20 | 180 | 157 | 81.89 | 100.00 |
| 20 | 20 | 20 | 200 | 45 | 82.22 | 100.00 |

various sizes having non-trivial autotopisms (Table 2). We find that computing the autotopism groups for partial Latin rectangles with non-trivial autotopisms indeed is more challenging. We also find that the exhaustive two-line graph refinement consistently outperforms the exhaustive natural refinement.

The random partial Latin rectangles in Table 2 are generated by starting with a PLR(20, 20, 20; 0), choosing a random autotopism, and filling in entries randomly in a way that is consistent with the autotopism. Afterwards, empty rows and columns are deleted, and the symbols are renumbered so each symbol in $[n]$ occurs.

We also test how effective the $\mathscr{P}_{\mathrm{SEI}}$ and $G(\mathscr{P}_1)$ partitions are (that is, applying the $N$ and $G$ refinements to the types and 1-part partitions, respectively) in detecting the triviality of Atop($L$). Here too we conduct experiments in PLR(8, 8, 8; $m$) and PLR(8, 9, 10; $m$). For each rectangle, we calculate $\mathscr{P}_{\mathrm{SEI}}$ and $G(\mathscr{P}_1)$. If at least two of the three partitions of the rows, columns and symbols are the partitions of singletons, then we conclude that Atop($L$) is trivial. The proportion of trivial-autotopism-group detection for the two methods is shown in Figs. 3 and 4.

We see that both $\mathscr{P}_{\mathrm{SEI}}$ and $G(\mathscr{P}_1)$ achieve near-100% detection for an intermediate number of entries, but struggle when the partial Latin rectangle is highly sparse or dense. However, $G(\mathscr{P}_1)$ is far more effective on denser partial Latin

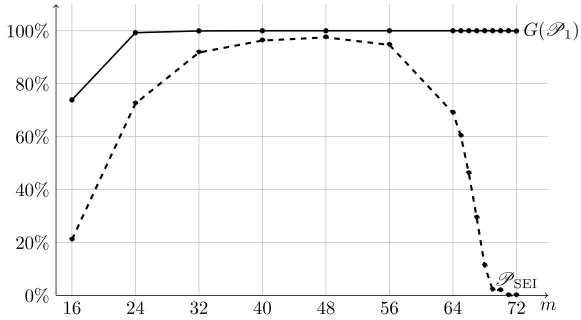



**Fig. 4.** Proportion of detection of the triviality of Atop($L$) for rectangles in PLR(8, 9, 10; $m$), by two refinement methods applied to 1000 random rectangles having trivial autotopism group, for each value of $m \in \{16, 24, 32, 40, 48, 56, 64, 65, \ldots, 72\}$.

rectangles than $\mathscr{P}_{\mathrm{SEI}}$. However, we note that setting the parameters $(r, s, n) = (8, 8, 8)$ particularly disadvantages the strong entry invariants.

In summary, the TLG refinement has an advantage over the natural refinement in two cases: (a) for very dense rectangles when $r = s = n$ (this includes Latin squares), and (b) when we only wish to apply one or two rounds of refinement, instead of exhausting the process.

## 8. Concluding remarks

In some cases, obtaining the orbit-system of partitions is not especially helpful in determining the autotopism group explicitly. For example, the partial Latin square of order nine

$$L = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \cdot & 9 & 4 & 2 & 8 & 7 & \cdot & 5 & 1 \\ \hline \cdot & \cdot & \cdot & 3 & \cdot & \cdot & 6 & \cdot & \cdot \\ \hline 8 & 5 & 7 & 4 & 1 & \cdot & \cdot & 9 & 2 \\ \hline 5 & \cdot & 1 & 8 & 7 & 9 & \cdot & 2 & 4 \\ \hline 9 & 2 & 5 & 7 & 4 & 1 & \cdot & 8 & \cdot \\ \hline 4 & 7 & 2 & 1 & 9 & 8 & \cdot & \cdot & 5 \\ \hline 1 & 4 & \cdot & 9 & 2 & 5 & \cdot & 7 & 8 \\ \hline \cdot & \cdot & \cdot & 6 & \cdot & \cdot & 3 & \cdot & \cdot \\ \hline 7 & 1 & 8 & 5 & \cdot & 2 & \cdot & 4 & 9 \\ \hline \end{array}$$

has an autotopism group of size 14, generated by the autotopisms $((28), \mathrm{Id}, (36))$ and $((1964357), (1582693), (1847925))$. All refinement methods described in this work yield the same row, column and symbol partitions: $\mathcal{P}_{\mathrm{row}} = \{\{1, 3, 4, 5, 6, 7, 9\}, \{2, 8\}\}$, $\mathcal{P}_{\mathrm{col}} = \{\{1, 2, 3, 5, 6, 8, 9\}, \{4\}, \{7\}\}$ and $\mathcal{P}_{\mathrm{sym}} = \{\{1, 2, 4, 5, 7, 8, 9\}, \{3, 6\}\}$, which describes the orbit-system of partitions, but they reduce the domain of the search by very little. There are still $7!^3 2!^2 = 512096256000$ candidates remaining. Some of the techniques used in Section 6 to avoid searching may be able to alleviate this problem, so we propose this as a future research direction.

One limitation of studying complexity asymptotically is that we implicitly assume at least one of the dimension parameters $r$, $s$ or $n$ is large. However, for small $r$, $s$ and $n$, the implicit constant hidden by the big-$O$ notation may be comparable to $\max(r, s, n)$. Moreover, in practice not much computational research is done on autotopisms of partial Latin rectangles that are large, such as with $\max(r, s, n) \geq 20$. (Although [41] is a fascinating exception.) We also only investigate worst-case complexity: we do not address average-case complexity and the complexity for sparse partial Latin rectangles, both of which may be asymptotically smaller.

An interesting research area is sub-linear time algorithms; with an input of size $N$, these algorithms perform their tasks in time $o(N)$. This leads to the question: Given a Latin square of order $n$ with a trivial autotopism group, can we computationally verify its autotopism group is trivial in time $o(n^2)$?

We also suggest the following research problems:

- Is there a characterization of the partial Latin rectangles for which the natural refinement, two-line graph refinement, or a combination of both refinements yields the orbits system of partitions?
- Suppose we acquired a system of partitions that is equal to or close to the orbit-system of partitions. What is the most efficient way to conduct the subsequent search?

In (6.2) we list an atomic Latin square; it is a rare but problematic case for benefiting from two-line partitions. Weakening the definition of "atomic" slightly, this Latin square also has the property that all of its two-column graphs are isomorphic, and all of its two-symbol graphs are isomorphic. It would be interesting to discover when this property occurs in partial Latin rectangles. Such "weak atomic" partial Latin rectangles are problematic for the same reason. A small example is the following:

| | | |
|---|---|---|
| 1 | 2 | · |
| 2 | · | 1 |
| · | 1 | 2 |

.

As an alternative to studying how autotopisms affect pairs of rows, it might also be worthwhile investigating how autotopisms affect the so-called quasigroup graphs [9] (a generalization of Cayley graphs to quasigroups) after adapting them to partial Latin rectangles with $r = n$. Effectively, we choose a set $S$ of columns, and construct a graph with vertex set $[n]$ and for each $i \in [n]$, we add a directed edge from $i$ to $L[i, j]$ for each $j \in S$. Another possibility is modifying this definition in order to drop the restriction $r = n$, and instead define a bipartite graph with vertex bipartition $[r] \cup [n]$, and directed edges $(i, L[i, j])$ for each $j \in S$. Either way, these are graphs that are invariant under autotopisms, and thus may be useful for computing autotopism groups.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Appendix. Glossary of symbols

| | |
|---|---|
| $\mathrm{PLR}(r, s, n)$ | Set of $r \times s$ partial Latin rectangles based on $[n]$. |
| $\mathrm{PLR}(r, s, n; m)$ | Set of $m$-entry members of $\mathrm{PLR}(r, s, n)$. |
| $L$ | Arbitrary partial Latin rectangle in $\mathrm{PLR}(r, s, n)$. |
| $\mathrm{Ent}(L)$ | Entry set of $L$. |
| $L^\pi$ | The conjugate of $L$ by $\pi \in S_3$. |
| $\mathrm{Atop}(L)$ | Autotopism group of $L$. |
| $\mathscr{P}$ | A system of partitions. |
| $\mathrm{SPart}(L)$ | The lattice of systems of partitions on $L$. |
| $\mathrm{SPart}_{\mathrm{Atop}}(L)$ | The lattice of systems of partitions on $L$ that are fixed by $\mathrm{Atop}(L)$. |
| $E(\mathscr{P})$ | Entry partition of $L$ that arises from $\mathscr{P}$. |
| $G(\mathscr{P})$ | Two-line graph refinement of $\mathscr{P}$ on $L$. |
| $G^\infty(\mathscr{P})$ | Exhaustive two-line graph refinement of $\mathscr{P}$ on $L$. |
| $N(\mathscr{P})$ | Natural refinement of $\mathscr{P}$ on $L$. |
| $N^\infty(\mathscr{P})$ | Exhaustive natural refinement of $\mathscr{P}$ on $L$. |
| $\mathscr{P}_{\mathrm{S}}(L)$ | System of partitions of singletons on $L$. |
| $\mathscr{P}_1(L)$ | System of 1-part partitions on $L$. |
| $\mathscr{P}_{\mathrm{O}}(L)$ | Orbit-system of partitions on $L$. |
| $\mathscr{P}_{\mathrm{T}}(L)$ | System of partitions defined by $L$'s types. |
| $\mathscr{P}_{\mathrm{SEI}}(L)$ | System of partitions defined by $L$'s strong entry invariants. |
| $\mathcal{R}_{\mathrm{row}}(L)$ | Two-row representation of $L$. |
| $\mathcal{R}_{\mathrm{col}}(L)$ | Two-column representation of $L$. |
| $\mathcal{R}_{\mathrm{sym}}(L)$ | Two-symbol representation of $L$. |
| $G_{\ell_1\ell_2}(L)$ | The $(\ell_1, \ell_2)$-line graph of $L$. |
| $\sigma(G)$ | Isomorphism class (IC) sequence of the two-line graph $G$. |
| $\sim_{\mathrm{row}}$ $(\sim_{\mathrm{col}}, \sim_{\mathrm{sym}})$ | Indicates two two-row (resp., two-column, two-symbol) graphs are isomorphic |
| $\sim_{\mathcal{P}}$ | Equivalence relation defined by a partition $\mathcal{P}$. |

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